MAT 733 — HOMEWORK 2

DUE ON WEDNESDAY 12 FEBRUARY

All rings are commutative with 1.

1. In $R = \mathbb{Z}[t]$ show that $\mathfrak{m} = (2,t)$ is a maximal ideal, and that $Q = (4,t)$ is a primary ideal which is not a power of a prime ideal.

2. Assume R is Noetherian and M is finitely generated. Prove that a prime ideal $\mathfrak{p}$ is associated to M if and only if there is an injective homomorphism $R/\mathfrak{p} \hookrightarrow M$.

3. Prove that "the rank of a finitely generated projective module is constant on connected components of the spectrum", as follows. See the footnotes for hints if desired.

(a) Prove that a finitely generated projective module P over a local ring $(R,\mathfrak{m})$ is free. [1]

(b) If P is f.g. projective and $P_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ of rank n, show that there exists $a \in R \setminus \mathfrak{p}$ such that P_a is free over R_a of rank n. [2]

(c) Let P be f.g. projective, and define $\Phi_P : \operatorname{Spec} R \longrightarrow \mathbb{N}$ by $\Phi_P(\mathfrak{p}) = \operatorname{rank} P_{\mathfrak{p}}$. Prove that Φ_P is continuous, with $\operatorname{Spec}(R)$ having the Zariski topology and $\mathbb{N}$ the discrete topology. [3]

(d) Conclude the statement in quotes above. [4]

4. Let $\mathbb{A}^n_k$ be affine n-space over a field k.

(a) Show that $\mathbb{A}^n_k$ is a Noetherian topological space w.r.t. the Zariski topology. (A topological space is Noetherian if it satisfies ACC on open sets.)

(b) Show that every variety in $\mathbb{A}^n_k$ is a finite union of irreducible varieties.

Hints:

[1] surjection $R^n \longrightarrow P$ splits, kill $\mathfrak{m}$, count vector-space dimensions and use NAK [if $M/\mathfrak{m}M = 0$ for finitely generated M then $M = 0$].

[2] lift the isomorphism $(R_{\mathfrak{p}})^n \cong P_{\mathfrak{p}}$ to a homomorphism [not iso!] $R^n \longrightarrow P$, and invert an element to kill the kernel and cokernel.

[3] the inverse image of $n \in \mathbb{N}$ contains an open set $D(a) = \operatorname{Spec}(R) \setminus V((a))$ for a as above.

[4] given that singletons in $\mathbb{N}$ are clopen