MAT 733 — HOMEWORK 1

DUE ON WEDNESDAY 29 JANUARY

All rings are commutative with 1.

1. Prove that a ring R is local if and only if the non-units of R form an ideal.

2. Let X be a topological space. Say that $U \subseteq X$ is *clopen* if it is both open and closed. (Yes, this is terrible.) Recall that X is *connected* iff the only clopen sets are X and $\emptyset$. Say that a ring R is connected if $\operatorname{Spec} R$ is connected (in the topological sense).

(i) Prove that R is connected iff there do not exist proper ideals I and J such that $I + J = R$ and $I \cap J = (0)$.

(ii) Prove that a local ring is connected.

(iii) Prove that a domain is connected.

(iv) Prove that $\mathbb{Z}/(6)$ is not connected.

(v) Let $e \in R$ be an idempotent (that is, $e^2 = e$). Show that $V((e))$ is clopen.

3. Prove that $\operatorname{Spec}(A \times B)$ is homeomorphic to the disjoint union of $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$. Discuss the relationship of this fact to part 2v above and the word “connected”.

4. Let $\operatorname{nil}(R)$ be the nilradical of R, that is, the set of nilpotent elements. Set $R_{\text{red}} = R/\operatorname{nil}(R)$. Prove that $\operatorname{Spec}(R)$ is homeomorphic to $\operatorname{Spec}(R_{\text{red}})$.

5. Let $\varphi\colon A \longrightarrow B$ be a ring homomorphism. Prove that the image of φ^* is dense in $\operatorname{Spec}(A)$ iff $\ker\varphi \subseteq \operatorname{nil}(A)$.