MAT 534 — HOMEWORK 8

DUE ON FRIDAY 21 MARCH

These first problems are from the Supplementary Exercises to Chapters 5–8.

1. (#8) Prove that S_4 is not isomorphic to $D_4 \oplus \mathbb{Z}_3$.

2. (#23) Exhibit four non-isomorphic groups of order 18. (Justify!)

3. (#30) Exhibit four non-isomorphic groups of order 66. (Justify!)

These next ones are from Chapter 9.

4. (#1) Prove that A_n is a normal subgroup of S_n.

5. (#9) If H is a subgroup of G such that $|G : H| = 2$, prove that H is normal in G. (Hint: partitions.)

6. (#11) Let $G = \mathbb{Z}_4 \oplus U(4)$. Let H be the cyclic subgroup generated by $(2,3)$, and let K be the cyclic subgroup generated by $(2,1)$. Observe that H and K are normal subgroups of G (do not hand this part in). Prove that $H \cong K$, but that $G/H \ncong G/K$.

7. (#12) Show that any factor group of an Abelian group is Abelian.

8. (#54) This problem defines the group of *quaternions*. Set

$$Q = \{\pm 1, \pm i, \pm j, \pm k\},$$

with multiplication defined by

$$i^2 = j^2 = k^2 = -1, \qquad -i = (-1)i, \qquad 1^2 = (-1)^2 = 1,$$

$$ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j.$$

(a) Construct a Cayley table for Q.

(b) Show that $H = \{1, -1\}$ is a normal subgroup of Q.

(c) Construct the Cayley table for Q/H. It's a group of order 4 – is it isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$?