MAT 631 — HOMEWORK 8

DUE ON THURSDAY 24 OCTOBER 2013

1. Use the fact that $M_{n\times n}(F)$ is an F-vector space of dimension n^2 to prove that for every $n\times n$ matrix A there exists a (monic) polynomial $f(x)=x^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0$, with $m\le n^2$ and the a_i in F, such that

$$f(A)\overset{\text{def}}{=}A^m+a_{m-1}A^{m-1}+\cdots+a_1A+a_oI_n=0.$$

2. Let Q be the quaternion group of order 8.

(a) Prove that Q is isomorphic to a subgroup of S_8.

(b) Prove that Q is *not* isomorphic to a subgroup of S_n for any $n\le 7$. (Hint: prove that in any action of Q on a set of size ≤ 7, the stabilizer of any point must contain the unique element of order 2, namely -1.)

(c) Prove that $C_2\times C_4$ is isomorphic to a subgroup of S_6. (So part (b) is really something special about Q, not just any group of order 8.)

3. Let G be a group, H a subgroup, and $a,\ b\in G$. Prove that $|b|=|aba^{-1}|$ and that $|H|=|aHa^{-1}|$.

4. Find all conjugacy classes for D_8.

5. Assume that $|G:Z(G)|=n$. Prove that every conjugacy class of G has size at most n. (Hint: Orbit-Stabilizer.)