

MAT 631 — HOMEWORK 7

DUE ON TUESDAY 15 OCTOBER 2013

1. Let W_1 and W_2 be subspaces of the vector space V . The *sum* of W_1 and W_2 is the set

$$W_1 + W_2 = \{w_1 + w_2 \mid w_i \in W_i\}.$$

It is a subspace; in particular it is the smallest subspace containing W_1 and W_2 . (You need not hand in a proof of either of the preceding assertions, but should check them for yourself.) Prove the following equality:

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

(You should address the case of infinite dimension, but it is relatively trivial. The main idea is to start with a finite basis of $W_1 \cap W_2$.)

2. Let P_n be the \mathbb{R} -vector space of polynomials $p(x) = a_0 + a_1x + \cdots + a_nx^n$ of degree at most n .
- (a) Let D denote the (formal) derivative $\frac{d}{dx}$, considered as a linear operator $P_n \rightarrow P_n$. Find $M_{\mathcal{B}}^{\mathcal{B}}(D)$, the matrix of D with respect to a convenient basis \mathcal{B} .
- (b) Let $\lambda \in \mathbb{R}$ be arbitrary and prove that $1, x - \lambda, (x - \lambda)^2, \dots, (x - \lambda)^n$ is a basis for P_n .
3. Set

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

- (a) Prove that \mathcal{B} is a basis for \mathbb{R}^3 .

- (b) Find the coordinate vector $[u]_{\mathcal{B}}$ of $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ with respect to \mathcal{B} .

- (c) Let \mathcal{C} be the standard basis for \mathbb{R}^3 . Find an invertible matrix P such that $P[v]_{\mathcal{B}} = [v]_{\mathcal{C}}$ for all $v \in \mathbb{R}^3$. (There are denominators in this answer.)

- (d) Define $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T(x, y, z) = (2x - z, y + 3z, x + y + z)$. Compute $M_{\mathcal{C}}^{\mathcal{C}}(T)$ and $M_{\mathcal{B}}^{\mathcal{B}}(T)$.