MAT 631 — HOMEWORK 3

DUE ON TUESDAY 17 SEPTEMBER 2013

1. Let A be an Abelian group and H a subgroup of A. Prove that A/H is Abelian. Give an example of a non-Abelian group G and a proper normal subgroup N of G such that G/N is Abelian.

2. Consider the additive group $\mathbb{Q}/\mathbb{Z}$.

(a) Show that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ contains precisely one representative q with $q \in [0,1)$.

(b) Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order but that there are elements of arbitrarily large order in $\mathbb{Q}/\mathbb{Z}$.

(c) Prove that $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the multiplicative group of roots of unity in $\mathbb{C}^\times$, i.e.

$$\mathbb{Q}/\mathbb{Z} \cong \left\{ z \in \mathbb{C}^\times \mid z^n = 1 \text{ for some } n \geq 1 \right\}.$$

3. Let $G = QD_{16}$ be the *quasidihedral group* of order 16, which has a presentation

$$QD_{16} = \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle.$$

(See exercise 11 in section 2.5 of D&F.)

(a) Verify that the center $Z(G)$ of G is cyclic, generated by σ^4. Set $\bar{G} = G/Z(G)$.

(b) Show that the order of $\bar{G}$ is 8.

(c) Exhibit each element of $\bar{G}$ in the form $\bar{\tau}^a \bar{\sigma}^b$ for some integers a and b.

(d) Find the order of each element of $\bar{G}$ above.

(e) Prove that $\bar{G} \cong D_8$.

4. (a) Prove that the intersection of any two normal subgroups of a group is again a normal subgroup.

(b) Assume that N is a normal subgroup of G, and $H < G$. Prove that $N \cap H$ is a normal subgroup of H.

(c) Let $H < K$ be subgroups of G, and prove: H is normal in K if and only if $K \subseteq N_G(H)$. Conclude that the normalizer of H is the largest subgroup of G in which H is normal.

5. (A classic) Prove that if $G/Z(G)$ is cyclic then G is Abelian. (Try it by yourself first, then see page 89 of D&F if you need a hint.)