MAT 631 — HOMEWORK 11

DUE ON TUESDAY 27 NOVEMBER

1. Let x and y be elements of a group, and define the *commutator* $[x,y] = x^{-1}y^{-1}xy$. Let G' be the subgroup generated by commutators. (Caution: this is not necessarily the *set of all* commutators, but is the smallest subgroup of G containing that set.) Prove that G/G' is the largest Abelian quotient of G, in the following sense: if N is a normal subgroup of G such that G/N is Abelian, then $G' \leq N$; and, if $G' \leq H \leq G$ then H is normal in G and G/H is Abelian.

2. Let S be a set, and consider the group presented as $\langle S \mid R \rangle$, where $R = \{[x,y] \mid x,y \in S\}$ is the set of all commutators of elements of S. This is called the *free Abelian group* on S, denoted $A = A_S$. Prove the universal mapping property of the free Abelian group: for any function $f : S \longrightarrow G$, where G is an Abelian group, there exists a unique group homomorphism $\varphi : A \longrightarrow G$ such that $\varphi(\overline{x}) = f(x)$ for every $x \in S$. Deduce that if $|S| = n$, then $A_S \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$ (n factors).

3. Let G be a group of order pq, where $p < q$ are primes. Prove that $G = PQ$, where P and Q are Sylow subgroups. If in addition $q \not\equiv 1 \bmod p$, conclude that G is Abelian. (Hint: consider $[x,y]$, where $x \in P$ and $y \in Q$.)