MAT 631 — HOMEWORK 8

DUE ON TUESDAY 23 OCTOBER

1. Suppose $v_1, \dots, v_n$ are eigenvalues for a linear operator $T\colon V \longrightarrow V$, and assume that the corresponding eigenvalues $c_1, \dots, c_n$ are distinct. Prove that $\{v_1, \dots, v_n\}$ is linearly independent. (Try $n = 2$ first to avoid being overwhelmed by notation.)

2. Let A be an $n \times n$ matrix with real entries. Assume that n is odd. Prove that A has at least one real eigenvalue. (Hint: calculus.)

3. Let A be an $n \times n$ matrix over a field F. Prove that there exists a polynomial $f(t) = t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0$ such that

$$f(A) = A^m + a_{m-1}A^{m-1} + \cdots + a_1 A + a_0 I = 0\,.$$

(Hint: consider the set $\{I, A, A^2, \dots\} \subseteq M_{n\times n}(F)$ and the fact that $\dim M_{n\times n}(F) = n^2$.) If B is similar to A, prove that $f(B) = 0$ as well.

4. Identify the coefficient of t^k in the product $\prod_{i=1}^{n}(t - c_i)$, for $k = 0, \dots, n$. Conclude that (a) the product of the eigenvalues of a square complex matrix A is $\det(A)$, and (b) the sum of the eigenvalues is $\mathrm{tr}(A)$. What's stopping you from concluding (a) and (b) above for an arbitrary field?