MAT 631 — HOMEWORK 5

DUE ON TUESDAY 2 OCTOBER

1. Let G be a group and let H, K be subgroups of finite index m and n, respectively. Prove that

$$\operatorname{lcm}(m,n) \le |G : H \cap K| \le mn\,.$$

Conclude: if $\gcd(|G:H|,\ |G:K|) = 1$, then $|G : H \cap K| = |G:H|\ |G:K|$.

2. Let $H \le K \le G$ be groups. Prove that $|G:H| = |G:K|\ |K:H|$. (Do not assume that G is finite nor that either subgroup is normal.)

3. Let G be a group and let H be a subgroup of index n.

(a) Prove that there is a homomorphism $\varphi\colon G \longrightarrow S_n$, with kernel a normal subgroup of H. (Hint: the action of G on left cosets of H by left-multiplication.)

(b) (bonus) If $n = p$ is the smallest prime factor of G, prove that H is normal in G. (Hint: apply 1^{st} Iso Thm and then Lagrange to φ above.)

4. Part (b) of this problem was on the August 2010 Prelim.

(a) Let A and B be subgroups of G such that $\gcd(|A|,\ |B|) = 1$. Prove that $A \cap B = \{1\}$.

(b) Let G, H, K be groups of orders 35, 60, and 42, respectively. Assume that there are homomorphisms $\varphi\colon G \longrightarrow H$ and $\psi\colon G \longrightarrow K$ such that $\ker\varphi \ne G$ and $\ker\psi \ne G$. Prove that $\ker\varphi \cap \ker\psi = \{1\}$.

5. (August 2011 Prelim) Let G be a group and let H, K be subgroups of G such that H is normal and $H \cap K = \{1\}$. Prove that $HK \cong H \times K$.