

MAT 631 — HOMEWORK 4

DUE ON TUESDAY 25 SEPTEMBER

1. Let A be an Abelian group and H a subgroup of A . Prove that A/H is Abelian. Give an example of a non-Abelian group G and a proper normal subgroup N of G such that G/N is Abelian.
2. Consider the additive group \mathbb{Q}/\mathbb{Z} .
 - (a) Show that every coset of \mathbb{Z} in \mathbb{Q} contains precisely one representative q with $q \in [0, 1)$.
 - (b) Show that every element of \mathbb{Q}/\mathbb{Z} has finite order but that there are elements of arbitrarily large order in \mathbb{Q}/\mathbb{Z} .
 - (c) Prove that \mathbb{Q}/\mathbb{Z} is isomorphic to the multiplicative group of roots of unity in \mathbb{C}^\times , i.e.

$$\mathbb{Q}/\mathbb{Z} \cong \{z \in \mathbb{C}^\times \mid z^n = 1 \text{ for some } n \geq 1\}.$$

3. Let $G = QD_{16}$ be the *quasidihedral group* of order 16, which has a presentation

$$QD_{16} = \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle.$$

(See exercise 11 in section 2.5 of D&F.)

- (a) Verify that the center $Z(G)$ of G is cyclic, generated by σ^4 . Set $\overline{G} = G/Z(G)$.
 - (b) Show that the order of \overline{G} is 8.
 - (c) Exhibit each element of \overline{G} in the form $\overline{\tau}^a \overline{\sigma}^b$ for some integers a and b .
 - (d) Find the order of each element of \overline{G} above.
 - (e) Prove that $\overline{G} \cong D_8$.
4.
 - (a) Prove that the intersection of any two normal subgroups of a group is again a normal subgroup.
 - (b) Assume that N is a normal subgroup of G , and $H < G$. Prove that $N \cap H$ is a normal subgroup of H .
 - (c) Let $H < K$ be subgroups of G , and prove: H is normal in K if and only if $K \subseteq N_G(H)$.
Conclude that the normalizer of H is the largest subgroup of G in which H is normal.
5. (A classic) Prove that if $G/Z(G)$ is cyclic then G is Abelian. (Try it by yourself first, then see page 89 of D&F if you need a hint.)