MAT 631 — HOMEWORK 3

DUE ON TUESDAY 18 SEPTEMBER

1. (a) Assume that a group H acts on a set X. Define $x \sim y$ if there is some $h \in H$ such that $h \cdot x = y$. Prove that "$\sim$" is an equivalence relation (page 3). The equivalence class containing x is called its *orbit*.

(b) Let G be a finite group, let H be a subgroup of G, and consider the action of H on G by left multiplication. Let $x \in G$ and let $H \cdot x = \{hx \mid h \in H\}$ be its orbit. Prove that $|H \cdot x| = |H|$ (so every orbit has the same size).

(c) Conclude *Lagrange's Theorem*: If G is a finite group and H is a subgroup, then $|H|$ divides $|G|$.

2. Let G be a group and X a subset of G. For an element $a \in G$, we write aXa^{-1} for the set of all elements of the form axa^{-1}, $x \in X$. The *normalizer* of X in G is

$$N_G(X) = \left\{a \in G \mid aXa^{-1} = X\right\}.$$

(a) Prove that $N_G(X)$ is a subgroup of G.

(b) Compute $N_{D_8}(\{1, \tau\})$.

(c) Compute $N_{S_3}(\{1, (1\ 2\ 3), (1\ 3\ 2)\})$.

3. Compute $Z(D_{2n})$. (The answer depends on the parity of n.)

4. Let G be a group and let H, K be subgroups of G.

(a) Prove that $H \cap K$ is a subgroup of G.

(b) Prove that $H \cup K$ is a subgroup of G only if one of H and K is contained in the other.

(c) Prove that the intersection of an arbitrary nonempty collection of subgroups of G is again a subgroup. (Do not assume that the collection is countable.)

(d) Let $x \in G$, and prove $\langle x \rangle = \bigcap_{x \in K} K$.

5. Compute the lattice of subgroups of C_{45}.

6. Prove that $(\mathbb{Z}/2^n\mathbb{Z})^\times$ is *not* cyclic for $n \geq 3$. (Exhibit two distinct subgroups of order 2, and explain why this is enough.)

7. Let $\varphi\colon G \longrightarrow H$ be a homomorphism. Prove that $\ker\varphi$ is a subgroup of G. Prove that φ is one-to-one if and only if $\ker\varphi$ is the trivial subgroup.