MAT 532 — HOMEWORK 2 — PROBLEM 2

2. Divide the interval $[0,1]$ into five equal subintervals, and apply the finite difference method to approximate a solution to the two-point boundary value problem

$$y''(t) = 8t, \qquad y(0) = 0, \qquad y(1) = 1.$$

Solution. Since we use 5 subintervals of $[0,1]$, we have $h = 1/5$, and the four interior grid points are $t_1 = 1/5$, $t_2 = 2/5$, $t_3 = 3/5$, and $t_4 = 4/5$. We also set $t_0 = 0$ and $t_5 = 1$ for consistency.

We want to approximate $y(t_i)$ for each i between 1 and 4, assuming $y''(t) = 8t$, $y(t_0) = 0$, and $y(t_5) = 1$. From lecture, we have an approximation

$$y''(t_i) \approx \frac{y(t_i - h) - 2y(t_i) + y(t_i + h)}{h^2}$$

for $i = 1,2,3,4$. If we set $y_i = y(t_i)$ (this is just for notational tidiness), this says

$$y_{i-1} - 2y_i + y_{i+1} \approx h^2 f(t_i).$$

Here $f(t) = 8t$. Write out each of the four equations:

$$\begin{array}{ccccccccccc}
0 & - & 2y_1 & + & y_2 & & & & & = & \frac{1}{25}\cdot 8\cdot\frac{1}{5} \\
 & & y_1 & - & 2y_2 & + & y_3 & & & = & \frac{1}{25}\cdot 8\cdot\frac{2}{5} \\
 & & & & y_2 & - & 2y_3 & + & y_4 & = & \frac{1}{25}\cdot 8\cdot\frac{3}{5} \\
 & & & & & & y_3 & - & 2y_4 \; + \; 1 & = & \frac{1}{25}\cdot 8\cdot\frac{4}{5}
\end{array}$$

Here I put in $y_0 = 0$ in the upper left corner, and $y_5 = 1$ in the lower right. The $\frac{1}{25}$ is h^2.

Translate to a matrix, and row-reduce:

$$\left[\begin{array}{cccc|c}
-2 & 1 & 0 & 0 & 8/125 \\
1 & -2 & 1 & 0 & 16/125 \\
0 & 1 & -2 & 1 & 24/125 \\
0 & 0 & 1 & -2 & -93/125
\end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & -7/125 \\
0 & 1 & 0 & 0 & -6/125 \\
0 & 0 & 1 & 0 & 11/125 \\
0 & 0 & 0 & 1 & 52/125
\end{array}\right]$$

Hence the approximate solution is $y_1 = \frac{-7}{125}$, $y_2 = \frac{-6}{125}$, $y_3 = \frac{11}{125}$, $y_4 = \frac{52}{125}$.