

# Tight Closure Theory and Characteristic $p$ Methods

MELVIN HOCHSTER

WITH AN APPENDIX BY GRAHAM J. LEUSCHKE

ABSTRACT. We give an introductory overview of the theory of tight closure, which has recently played a primary role among characteristic- $p$  methods. We shall see that such methods can be used even when the ring contains a field of characteristic 0.

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## Introduction

The theory of tight closure has recently played a primary role among commutative algebraic methods in characteristic  $p$ . We shall see that such methods can be used even when the ring contains a field of characteristic 0.

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Unless otherwise specified, the rings that we consider here will be Noetherian rings  $R$  containing a field. Frequently, we restrict, for simplicity, to the case of domains finitely generated over a field  $K$ . The theory of tight closure exists in much greater generality. For the development of the larger theory and its applications, and for discussion of related topics such as the existence of big Cohen–Macaulay algebras, we refer the reader to the joint works by Hochster and Huneke listed in the bibliography, to [Hochster 1994a; 1994b; 1996], to the expository accounts [Bruns 1996; Huneke 1996; 1998], and to the appendix to this paper by Graham Leuschke.

Here, in reverse order, are several of the most important reasons for studying tight closure theory, which gives a closure operation on ideals and on submodules. We focus mostly on the case of ideals here, although there is some discussion of modules. We shall elaborate on the themes brought forth in the list below in the sequel.

11. Tight closure can be used to shorten difficult proofs of seemingly unrelated results. The results turn out to be related after all. Often, the new results are stronger than the original results.
10. Tight closure provides algebraic proofs of several results that can otherwise be proved only in equal characteristic 0, and whose original proofs depended on analytic techniques.
9. In particular, tight closure can be used to prove the Briançon–Skoda theorem on integral closures of ideals in regular rings.
8. Likewise, tight closure can be used to prove that rings of invariants of linearly reductive algebraic groups acting on regular rings are Cohen–Macaulay.
7. Tight closure can be used to prove several of the local homological conjectures.
6. Tight closure can be used to “control” certain cohomology modules: in particular, one finds that the Jacobian ideal kills them.
5. Tight closure implies several vanishing theorems that are very difficult from any other point of view.
4. Tight closure controls the behavior of ideals when they are expanded to a module-finite extension ring and then contracted back to the original ring.
3. Tight closure controls the behavior of certain colon ideals involving systems of parameters.
2. Tight closure provides a method of compensating for the failure of ambient rings to be regular.
1. If a ring is already regular, the tight closure is very small: it coincides with the ideal (or submodule). This gives an extraordinarily useful test for when an element is in an ideal in regular rings.

One way of thinking about many closure operations is to view them as arising from necessary conditions for an element to be in an ideal. If the condition fails, the element is not in the ideal. If the condition is not both necessary and

sufficient, then when it holds, the element might be in the ideal, but it may only be in some larger ideal, which we think of as a kind of closure.

Tight closure in positive characteristic can be thought of as arising from such a necessary but not sufficient condition for ideal membership. One of the reasons that it is so useful for proving theorems is that in some rings, the condition is both necessary and sufficient. In particular, that is true in regular rings. In consequence, many theorems can be proved about regular rings that are rather surprising. They have the following nature: one can see that in a regular ring a certain element is “almost” in an ideal. Tight closure permits one to show that the element actually is in the ideal. This technique works like magic on several major results that seemed very difficult before tight closure came along.

One has to go to some considerable trouble to get a similar theory working in rings that contain the rationals, but this has been done, and the theory works extremely well for “nice” Noetherian rings like the ones that come up in algebraic and analytic geometry.

It is still a mystery how to construct a similar theory for rings that do not contain a field. This is not a matter of thinking about anything pathological. Many conjectures could be resolved if one had a good theory for domains finitely generated as algebras over the integers.

Before proceeding to talk about tight closure, we give some examples of necessary and/or sufficient conditions for membership in an ideal. The necessary conditions lead to a kind of closure.

**(1)** A necessary condition for  $r \in R$  to be in the ideal  $I$  is that the image of  $r$  be in  $IK$  for every homomorphism of  $R$  to a field  $K$ . This is not sufficient: the elements that satisfy the condition are precisely the elements with a power in  $I$ , the *radical* of  $I$ .

**(2)** A necessary condition for  $r \in R$  to be in the ideal  $I$  is that the image of  $r$  be in  $IV$  for every homomorphism of  $R$  to a valuation ring  $V$ . This is not sufficient: the elements that satisfy the condition are precisely the elements in  $\bar{I}$ , the *integral closure* of  $I$ . If  $R$  is Noetherian, one gets the same integral closure if one only considers Noetherian discrete valuation rings  $V$ . There are many alternative definitions of integral closure.

**(3)** If  $R$  has positive prime characteristic  $p$  let  $S_e$  denote  $R$  viewed as an  $R$ -algebra via the  $e$ -th iteration  $F^e$  of the Frobenius endomorphism  $F$  (thus,  $S_e = R$ , but the structural homomorphism  $R \rightarrow S_e = R$  sends  $r$  to  $r^{p^e}$ ). A necessary condition that  $r \in I$  is that for some integer  $e$ ,  $r^{p^e} \in IS_e$ . Note that when  $S_e$  is identified with  $R$ ,  $IS_e$  becomes the ideal generated by all elements  $i^{p^e}$  for  $i \in I$ . This ideal is denoted  $I^{[p^e]}$ . This condition is not sufficient for membership in  $I$ . The corresponding closure operation is the Frobenius closure  $I^F$  of  $I$ : it consists of all elements  $r \in R$  such that  $r^{p^e} \in I^{[p^e]}$  for some nonnegative integer  $e$ . (Once

this holds for once choice of  $e$ , it holds for all larger choices.) For example, in

$$K[x, y, z] = K[X, Y, Z]/(X^3 + Y^3 + Z^3),$$

if  $K$  has characteristic 2 (quite explicitly, if  $K = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ ) then with  $I = (x, y)$ , we have  $z^2 \in I^F - I$ . In fact,  $(z^2)^2 \in I^{[2]} = (x^2, y^2)$  here, since  $z^4 = z^3z = -(x^3 + y^3)z \in (x^2, y^2)$ .

Finally, here is a test for ideal membership that is sufficient but *not* necessary. It was used in the first proof of the Briançon–Skoda theorem, and so we mention it, although easier proofs by analytic methods are available now.

**(4) Skoda's analytic criterion.** Let  $\Omega$  be a pseudoconvex open set in  $\mathbb{C}^n$  and  $\phi$  a plurisubharmonic function<sup>1</sup> on  $\Omega$ . Let  $f$  and  $g_1, \dots, g_k$  be holomorphic functions on  $\Omega$ . Let  $\gamma = (|g_1|^2 + \dots + |g_k|^2)^{1/2}$ . Let  $X$  be the set of common zeros of the  $g_j$ . Let  $d = \max\{n, k-1\}$ . Let  $\lambda$  denote Lebesgue measure on  $\mathbb{C}^n$ . Skoda's criterion asserts that *if either*

$$\int_{\Omega-X} \frac{|f|^2}{\gamma^{2\alpha d+2}} e^{-\phi} d\lambda < +\infty,$$

for some real  $\alpha > 1$ , or

$$\int_{\Omega-X} \frac{|f|^2}{\gamma^{2d}} (1 + \Delta \log(\gamma)) e^{-\phi} d\lambda < +\infty,$$

then there exist  $h_1, \dots, h_k$  holomorphic on  $\Omega$  such that  $f = \sum_{j=1}^k h_j g_j$ . Hilbert's Nullstellensatz states that if  $f$  vanishes at the common zeros of the  $g_j$  then  $f \in \text{Rad } I$  where  $I = (g_1, \dots, g_k)$ . The finiteness of any of the integrals above conveys the stronger information that, in some sense,  $f$  is “small” whenever all the  $g_j$  are “small” (or the integrand will be too “large” for the integral to converge), and we get the stronger conclusion that  $f \in I$ .

## 1. Reasons for Thinking About Tight Closure

We give here five results valid in any characteristic (i.e., over any field) that can be proved using tight closure theory. The tight closure proofs are remarkably simple, at least in the main cases. The terminology used in the following closely related theorems is discussed briefly after their statements.

**THEOREM 1.1 (HOCHSTER AND ROBERTS).** *Let  $S$  be a regular ring that is an algebra over the field  $K$ , and let  $G$  be a linearly reductive algebraic group over  $K$  acting on  $S$ . Then the ring of invariants  $R = S^G$  is a Cohen–Macaulay ring.*

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<sup>1</sup>We won't explain these terms from complex analysis here: the definitions are not so critical for us, because in the application to the Briançon–Skoda theorem, which we discuss later, we work in the ring of germs of holomorphic functions at, say, the origin in complex  $n$ -space,  $\mathbb{C}^n$  ( $\cong$  convergent power series  $\mathbb{C}\{z_1, \dots, z_n\}$ ) — we can pass to a smaller, pseudoconvex neighborhood; likewise,  $\phi$  becomes unimportant.

**THEOREM 1.1° (HOCHSTER AND HUNEKE).** *If  $R$  is a direct summand (as an  $R$ -module) of a regular ring  $S$  containing a field, then  $R$  is Cohen–Macaulay.*

Theorem 1.1° implies Theorem 1.1. Both apply to many examples from classical invariant theory. Recall that an algebraic group (i.e., a Zariski closed subgroup of  $GL(n, K)$ ) is called *linearly reductive* if every representation is completely reducible. In characteristic 0, these are the same as the reductive groups and include finite groups, products of  $GL(1, K)$  (algebraic tori), and semi-simple groups. Over  $\mathbb{C}$  such a group is the complexification of compact real Lie group. A key point is that when a linearly reductive algebraic group acts on a  $K$ -algebra  $S$ , if  $S^G$  is the ring of invariants or fixed ring  $\{s \in S : g(s) = s \text{ for all } g \in G\}$  there is a canonical retraction map  $S \rightarrow S^G$ , called the *Reynolds operator*, that is  $S^G$ -linear. Thus,  $R = S^G$  is a direct summand of  $S$  as an  $R$ -module.

In particular, if  $S$  is a polynomial ring over a field  $K$  and  $G$  is a linearly reductive linear algebraic group acting on  $S_1$ , the vector space of 1-forms of  $S$ , and, hence, all of  $S$  (the action should be an appropriate one, i.e., determined by a  $K$ -morphism of  $G$  into the automorphisms of the vector space  $S_1$ ), then the fixed ring  $S^G$  is a Cohen–Macaulay ring  $R$ . What is a Cohen–Macaulay ring? The issue is local: for a local ring the condition means that some (equivalently, every) system of parameters is a regular sequence. In the graded case the Cohen–Macaulay condition has the following pleasant interpretation: when  $R$  is represented as a finitely generated module over a graded polynomial subring  $A$ ,  $R$  is free over  $A$ . This is a very restrictive and useful condition on  $R$ , especially in higher dimension. The Cohen–Macaulay condition is very important in intersection theory. Notice that since moduli spaces are frequently constructed as quotients of smooth varieties by actions of reductive groups, Theorem 1.1 implies the Cohen–Macaulay property for many moduli spaces.

Theorem 1.1 was first proved by a complicated reduction to characteristic  $p > 0$  [Hochster and Roberts 1974]. Boutot [1987] gave a shorter proof for affine algebras in characteristic 0 using resolution of singularities and the Grauert–Riemenschneider vanishing theorem. The tight closure proof of Theorem 1.1° is the simplest in many ways.

**THEOREM 1.2 (BRIANÇON–SKODA THEOREM).** *Let  $R$  be a regular ring and  $I$  an ideal of  $R$  generated by  $n$  elements. Then  $\bar{I}^n \subseteq I$ .*

We gave one characterization of what  $u \in \bar{J}$  means earlier. It turns out to be equivalent to require that there be an equation

$$u^h + j_1 u^{h-1} + \cdots + j_h = 0$$

such that every  $j_t \in J^t$ ,  $1 \leq t \leq h$ . We shall give a third characterization later.

Theorem 1.2 was first proved by analytic techniques; compare (4) on page 184. See [Skoda and Briançon 1974; Skoda 1972]: in the latter paper the analytic criteria needed were proved. The first algebraic proofs were given in [Lipman

and Teissier 1981] (for a very important special case) and [Lipman and Sathaye 1981]. There are several instances in which tight closure can be used to prove results that were first proved either by analytic techniques or by results like the Kodaira vanishing theorem and related characteristic 0 vanishing theorems in algebraic geometry. See [Huneke and Smith 1997] for a discussion of the connection with the Kodaira vanishing theorem.

**THEOREM 1.3 (EIN–LAZARSFELD–SMITH COMPARISON THEOREM).** *Let  $P$  be a prime ideal of codimension  $h$  in a regular ring. Then  $P^{(hn)} \subseteq P^n$  for every integer  $n$ .*

This was most unexpected. The original proof, valid in characteristic 0, ultimately depends on resolution of singularities and deep vanishing theorems, as well as a theory of asymptotic multiplier ideals. See [Ein et al. 2001]. The tight closure proof in [Hochster and Huneke 2002] permits one to extend the results to characteristic  $p$  as well as recovering the characteristic 0 result. There are other connections between tight closure theory and the theory of multiplier ideals: see [Smith 2000; Hara 2001; Hara and Yoshida 2003].

**THEOREM 1.4 (HOCHSTER AND HUNEKE).** *Let  $R$  be a reduced equidimensional finitely generated  $K$ -algebra, where  $K$  is algebraically closed. Let  $f_1, \dots, f_h$  be elements of  $R$  that generate an ideal  $I$  of codimension (also called height)  $h$  mod every minimal prime of  $R$ . Let  $J$  be the Jacobian ideal of  $R$  over  $K$ . Then  $J$  annihilates the Koszul cohomology  $H^i(f_1, \dots, f_h; R)$  for all  $i < h$ , and hence the local cohomology  $H_I^i(R)$  for  $i < h$ .*

This result is a consequence of phantom homology theory, test element theory for tight closure, and the Lipman–Sathaye Jacobian theorem [Lipman and Sathaye 1981], all of which we will describe eventually. If

$$R \cong K[x_1, \dots, x_n]/(f_1, \dots, f_m)$$

has codimension  $r$  in  $\mathbb{A}_K^n$ , then  $J$  is the ideal of  $R$  generated by the images of the size  $r$  minors of the Jacobian matrix  $(\partial f_j / \partial x_i)$ , and defines the non-smooth (over  $K$ ) locus in  $\text{Spec } R$ . The ideal  $J \subseteq R$  turns out to be independent of which presentation of  $R$  one chooses.

Here is a more geometrically flavored corollary.

**COROLLARY 1.5.** *Let  $R$  be a finitely generated graded domain of dimension  $n+1$  over an algebraically closed field  $K$ , so that  $X = \text{Proj}(R)$  is a projective variety of dimension  $n$  over an algebraically closed field  $K$ . Let  $g$  denote a homogeneous element of the Jacobian ideal  $J \subseteq R$  of degree  $d$  (so that  $g$  gives a global section of  $\mathcal{O}_X(d)$ ). Then for  $1 \leq j \leq n-1$ , the map  $H^j(X, \mathcal{O}_X(t)) \rightarrow H^j(X, \mathcal{O}_X(t+d))$  induced by multiplication by  $g$  is 0.*

The reason this follows from Theorem 1.4 is that for  $j \geq 1$ , if we let  $M = \bigoplus_{t \in \mathbb{Z}} H^j(X, \mathcal{O}_X(t))$ , then  $M$  is isomorphic (as an  $R$ -module) with  $H_m^{j+1}(R)$

which may be viewed as an  $R$ -module. We may replace  $m$  by the ideal generated by a homogeneous system of parameters, since the two have the same radical. Then Theorem 1.4 implies that the Jacobian ideal of  $R$  kills  $M$  for  $1 \leq j \leq n-1$ . See also Corollary 8.3.

## 2. The Definition of Tight Closure in Positive Characteristic

One of our guidelines towards a heuristic feeling for when an element  $u$  of a Noetherian ring  $R$  should be viewed as “almost” in an ideal  $I \subseteq R$  will be this: if  $R$  has a module-finite extension  $S$  such that  $u \in IS$  then  $u$  is “almost” in  $I$ .

Notice that if  $R$  is a normal domain (i.e., integrally closed in its field of fractions) containing the rational numbers and  $S$  is a module-finite extension, then  $IS \cap R = I$ , so that for normal rings containing  $\mathbb{Q}$  we are not allowing any new elements into the ideal. One can see this as follows. By first killing a minimal prime ideal of  $S$  disjoint from  $R - \{0\}$  we may assume that  $S$  is a domain. Let  $L \rightarrow L'$  be the corresponding finite algebraic extension of fraction fields, and suppose it has degree  $d$ . Let  $\text{tr}_{L'/L}$  denote field trace. Then  $\frac{1}{d} \text{tr}_{L'/L} : S \rightarrow R$  gives an  $R$ -linear retraction when  $R$  is normal. This implies that  $IS \cap R = I$  for every ideal  $I$  of  $R$ . (We only need the invertibility of the single integer  $d$  in  $R$  for this argument.)

The situation for normal domains of positive characteristic is very different, where it is an open question whether the elements that are “almost” in an ideal in this sense may coincide with the tight closure in good cases. Our definition of tight closure may seem unrelated to the notion above at first, but there is a close connection.

For simplicity we start with the case of ideals in Noetherian domains of characteristic  $p > 0$ . Recall that in characteristic  $p$  the Frobenius endomorphism  $F = F_R$  on  $R$  maps  $r$  to  $r^p$ , and is a ring endomorphism. When  $R$  is reduced, we denote by  $R^{1/p^e}$  the ring obtained by adjoining  $p^e$ -th roots for all elements of  $R$ : it is isomorphic to  $R$ , using the  $e$ -th iterate of its Frobenius endomorphism with the image restricted to  $R$ . Recall that in a ring of positive characteristic  $p$ , when  $q = p^e$ , we denote by  $I^{[q]}$  the ideal of  $R$  generated by all  $q$ -th powers of elements of  $I$ . It is easy to see that this ideal is generated by  $q$ -th powers of generators of  $I$ . Notice that it is *much* smaller, typically, than the ordinary power  $I^q$ .  $I^q$  is generated by *all monomials* of degree  $q$  in the generators of  $I$ , not just  $q$ -th powers of generators.

**DEFINITION 2.1.** Let  $R$  be a Noetherian domain of characteristic  $p > 0$ , let  $I$  be an ideal of  $R$ , and let  $u$  be an ideal of  $R$ . We say that  $u \in R$  is in the *tight closure*  $I^*$  of  $I$  in  $R$  if there exists an element  $c \in R - \{0\}$  such that for all sufficiently large  $q = p^e$ , we have  $cu^q \in I^{[q]}$ .

It is equivalent in the definition above to say “for all  $q$ ” instead of “for all sufficiently large  $q$ ”. We discuss why this condition should be thought of as

placing  $u$  “almost” in  $I$  in some sense. Let  $I = (f_1, \dots, f_h)R$ . Note that for every large  $q = p^e$  one has

$$cu^q = r_{1q}f_1^q + \cdots + r_{hq}f_h^q$$

and if we take  $q$ -th roots we have

$$c^{1/q}u = r_{1q}^{1/q}f_1 + \cdots + r_{hq}^{1/q}f_h,$$

an equation that holds in the ring  $S_q = R[c^{1/q}, r_{iq}^{1/q} : 1 \leq i \leq h]$ .  $S$  is a module-finite extension of  $R$ . But this is not quite saying that  $u$  is in  $IS$ : rather, it says that  $c^{1/q}u$  is in  $IS$ . But for very large  $q$ , for heuristic purposes, one may think of  $c^{1/q}$  as being close to 1: after all, the exponent is approaching 0. Thus,  $u$  is multiplied into  $IS_q$  in a sequence of module-finite extensions by elements that are getting closer and closer to being a unit, in a vague heuristic sense. This may provide some motivation for the idea that elements that are in the tight closure of an ideal are “almost” in the ideal.

It is ironic that tight closure is an extremely useful technique for proving theorems about regular rings, because it turns out that in regular rings the tight closure of any ideal  $I$  is simply  $I$  itself. In some sense, the reason that tight closure is so useful in regular rings is that it gives a criterion for being in an ideal that, on the face of it, is considerably weaker than being in the ideal. We shall return to this point later.

We may extend the definition to Noetherian rings  $R$  of positive prime characteristic  $p$  that are not necessarily integral domains in one of two equivalent ways:

- (1) Define  $u$  to be in  $I^*$  if the image of  $u$  in  $R/P$  is in the tight closure of  $I(R/P)$  in  $R/P$  for every minimal prime  $P$  of  $R$ .
- (2) Define  $u$  to be in  $I^*$  if there is an element  $c \in R$  and not in any minimal prime of  $R$  such that  $cu^q \in I^{[q]}$  for all  $q = p^e \gg 0$ .

### 3. Basic Properties of Tight Closure and the Briançon–Skoda Theorem

The following facts about tight closure in a Noetherian ring  $R$  of positive prime characteristic  $p$  are reasonably easy to verify from the definition.

- (a) For any ideal  $I$  of  $R$ ,  $(I^*)^* = I^*$ .
- (b) For any ideals  $I \subseteq J$  of  $R$ ,  $I^* \subseteq J^*$ .

We shall soon need the following characterization of integral closure of ideals in Noetherian domains.

FACT 3.0. *Let  $R$  be a Noetherian domain and let  $J$  be an ideal. Then  $u \in R$  is in  $\bar{J}$  if and only if for some  $c \in R - \{0\}$  and every integer positive integer  $n$ ,  $cu^n \in I^n$ . It suffices if  $cu^n \in I^n$  for infinitely many values of  $n$ .*

Comparing this with the definition of tight closure and using the fact that  $I^{[q]} \subseteq I^q$  for all  $q = p^e$ , we immediately get

- (c) For any ideal  $I$  of  $R$ ,  $I^* \subseteq \bar{I}$ . In particular,  $I^*$  is contained in the radical of  $I$ .

Less obvious is the following theorem that we will prove later.

**THEOREM 3.1.** *If  $R$  is regular, every ideal of  $R$  is tightly closed.*

Assuming this fact for a moment, we can prove the following result:

**THEOREM 3.2 (TIGHT CLOSURE FORM OF THE BRIANÇON–SKODA THEOREM IN CHARACTERISTIC  $p$ ).** *Let  $I = (f_1, \dots, f_n)R$  be an ideal of a regular ring  $R$  of characteristic  $p > 0$ . Then  $\bar{I}^n \subseteq I$ . When  $R$  is not necessarily regular, it is still true that  $\bar{I}^n \subseteq I^*$ .*

**PROOF.** Assuming Theorem 3.1 for the moment, we need only check the final assertion. It suffices to work modulo each minimal prime of  $R$  in turn, so we may assume that  $R$  is a domain. Then  $u \in \bar{I}^n$  implies that for some nonzero  $c$ ,  $cu^m \in (I^n)^m$  for all  $m$ . Restricting  $m = q = p^e$  we find that  $cu^m \in I^{nq} \subseteq I^{[q]}$  for all  $q$ , since a monomial in  $n$  elements of degree  $nq$  must have a factor in which one of the elements is raised to the  $q$ -th power.  $\square$

Why is every ideal in a regular ring tightly closed? We first need the following:

**FACT 3.3.** *If  $R$  is regular of positive characteristic  $p$ , the Frobenius endomorphism is flat.*

**PROOF.** The issue is local on  $R$ . In the local case it suffices to prove it for the completion  $\widehat{R}$  because  $R \rightarrow \widehat{R}$  is faithfully flat. We have therefore reduced to considering the case  $R \cong K[[x_1, \dots, x_n]]$ . The Frobenius map is then isomorphic with the ring inclusion  $K^p[[x_1^p, \dots, x_n^p]] \subseteq K[[x_1, \dots, x_n]]$ . Letting  $K^p = k$ , we may factor this map as  $k[[x_1^p, \dots, x_n^p]] \subseteq k[[x_1, \dots, x_n]] \subseteq K[[x_1, \dots, x_n]]$ . The first extension is free on the monomials  $x_1^{h_1} \cdots x_n^{h_n}$  with  $0 \leq h_i < p$  for all  $i$ . The flatness of the second map (for any field inclusion  $k \subseteq K$ ) may be seen as follows: since  $K$  is flat (in fact, free) over  $k$ ,  $K[[x_1, \dots, x_n]]$  is flat over  $k[[x_1, \dots, x_n]]$ . This is preserved when we localize at the maximal ideal generated by the  $x$ 's in the larger ring and its contraction (also generated by the  $x$ 's) to the smaller ring. Finally, it is further preserved when we complete both local rings.  $\square$

Recall that for an ideal  $I$  of  $R$  and element  $u \in R$ ,  $I : u = \{r \in R : ur \in I\}$ . This may be thought of as the annihilator in  $R$  of the image of  $u$  in  $R/I$ .

**FACT 3.4.** *If  $f : R \rightarrow S$  is flat,  $I \subseteq R$  and  $u \in R$ , then  $IS :_S f(u) = (I :_R u)S$ .*

To see why, note the exact sequence  $(I : u)/I \rightarrow R/I \rightarrow R/I$  where the map is multiplication by  $u$ . Applying  $S \otimes_R$  preserves exactness, from which the stated result follows.

COROLLARY 3.5. *If  $R$  is regular of positive characteristic,  $I$  is any ideal, and  $u \in R$ , then  $I^{[q]} : u^q = (I : u)^{[q]}$  for all  $q = p^e$ .*

The point is that since  $F : R \rightarrow R$  is flat, so is its  $e$ -th iterate  $F^e$ . If  $S$  denotes  $R$  viewed as an  $R$ -algebra via  $F^e$  then  $IS = I^{[p^e]}$  when we “remember” that  $S$  is  $R$ . With this observation, Corollary 3.5 follows from Fact 3.4.

PROOF OF THEOREM 3.1. We can reduce to the case where  $R$  is a domain. If  $c \neq 0$  and  $cu^q \in I^{[q]}$  for all  $q = p^e$ , then  $c \in \bigcap_q I^{[q]} : u^q = \bigcap_q (I : u)^{[q]} \subseteq \bigcap_q (I : u)^q$ . Since the intersection is not 0, we must have that  $I : u = R$ , i.e., that  $u \in R$ .  $\square$

We also mention here the very useful fact that tight closure captures contracted extensions from module-finite extensions.

THEOREM 3.6. *Let  $S$  be a domain module-finite over  $R$  and let  $I$  be an ideal of  $R$ . Then  $IS \cap R \subseteq I^*$ .*

PROOF.  $S$  can be embedded in a finitely generated free  $R$ -module. One of the projection maps back to  $R$  will be nonzero on the identity element of  $S$ . That is, there is an  $R$ -linear map  $f : S \rightarrow R$  that sends  $1 \in S$  to  $c \in R - \{0\}$ . If  $u \in IS \cap R$ , then  $u^q \in I^{[q]}S$  for all  $q$ . Applying  $f$  to both sides yields that  $cu^q \in I^q$ .  $\square$

Although we have not yet given the definitions the analogous fact holds for submodules of free modules, and can even be formulated for arbitrary submodules of arbitrary modules.

#### 4. Direct Summands of Regular Rings are Cohen–Macaulay

Elements  $x_1, \dots, x_n$  in a ring  $R$  are called a *regular sequence* on an  $R$ -module  $M$  if  $(x_1, \dots, x_n)M \neq M$  and  $x_{i+1}$  is not a zerodivisor on  $M/(x_1, \dots, x_i)M$ ,  $0 \leq i < n$ . A sequence of indeterminates in a polynomial or formal power series ring  $R$ , with  $M = R$  (or a nonzero free  $R$ -module) is an example. We shall make use of the following fact:

FACT 4.1. *Let  $A$  be a polynomial ring over a field  $K$ , say  $A = K[x_1, \dots, x_d]$  or let  $A$  be a regular local ring in which  $x_1, \dots, x_d$  is a minimal set of generators of the maximal ideal. Then a finitely generated nonzero  $A$ -module  $M$  (assumed graded in the first case) is  $A$ -free if and only if  $x_1, \dots, x_d$  is a regular sequence on  $M$ . Thus, a module-finite extension ring  $R$  (graded if  $A$  is a polynomial ring) of  $A$  is Cohen–Macaulay if and only if  $x_1, \dots, x_d$  is a regular sequence on  $R$ .*

The following two lemmas make the connection between tight closure and the Cohen–Macaulay property.

PROPOSITION 4.2. *Let  $S$  be a module-finite domain extension of the domain  $R$  (torsion-free is sufficient) and let  $x_1, \dots, x_d$  be a regular sequence in  $R$ . Suppose  $0 \leq k < d$  and let  $I = (x_1, \dots, x_k)R$ . Then  $IS :_S x_{k+1} \subseteq (IS)^*$  in  $S$ .*

Thus, if every ideal of  $S$  is tightly closed, and  $x_1, \dots, x_d$  is a regular sequence in  $R$ , it is a regular sequence in  $S$ .

PROOF. Because  $S$  is a torsion-free  $R$ -module there is an element  $c$  of  $R - \{0\}$  that multiplies  $S$  into an  $R$ -free submodule  $G \cong R^h$  of  $S$ . (This is really all we need about  $S$ .) Suppose that  $ux_{k+1} \in IS$ . Raise both sides to the  $q = p^e$  power to get  $u^qx_{k+1}^q \in I^{[q]}S$ . Multiply by  $c$  to get  $(cu^q)x_{k+1}^q \in I^{[q]}G$ . Because the  $x_j$  form a regular sequence on  $G$ , so do their  $q$ -th powers, and we find that  $cu^q \in I^{[q]}S = (IS)^{[q]}$ . Since this holds for all  $q = p^e$ , we are done.  $\square$

**PROPOSITION 4.3.** *Let  $R$  be a domain module-finite over a regular local ring  $A$  or  $\mathbb{N}$ -graded and module-finite over a polynomial ring  $A$ . Suppose that  $R$  is a direct summand of a regular ring  $S$  as an  $R$ -module. Then  $R$  is Cohen–Macaulay (i.e.,  $A$ -free).*

PROOF. Let  $x_1, \dots, x_d$  be as in Fact 4.1. The result comes down to the assertion that  $x_1, \dots, x_d$  is a regular sequence on  $R$ . By Proposition 4.2, it suffices to show that every ideal of  $R$  is tightly closed. But if  $J$  is an ideal of  $R$  and  $u \in R$  is in  $J^*$ , then it is clear that  $u \in (JS)^* = JS$ , since  $S$  is regular, and so  $u \in JS \cap R = J$ , because  $R$  is a direct summand of  $S$ .  $\square$

Pushing this idea a bit further, one gets a full proof of Theorem 1.1. We need to extend the notion of tight closure to equal characteristic 0, however. This is tackled in Section 6.

## 5. The Ein–Lazarsfeld–Smith Comparison Theorem

We give here the characteristic- $p$  proof of Theorem 1.3, and we shall even allow radical ideals, with  $h$  taken to be the largest height of any minimal prime. For a prime ideal  $P$ ,  $P^{(N)}$ , the  $N$ -th symbolic power, is the contraction of  $P^NR_P$  to  $R$ . When  $I$  is a radical ideal with minimal primes  $P_1, \dots, P_k$  and  $W = R - \bigcup_j P_j$ , we may define  $P^{(N)}$  either as  $\bigcap_j P_j^{(N)}$  or as the contraction of  $I^N(W^{-1}R)$  to  $R$ .

Suppose that  $I \neq (0)$  is radical ideal. If  $u \in I^{(hn)}$ , then for every  $q = p^e$  we can write  $q = an + r$  where  $a \geq 0$  and  $0 \leq r \leq n-1$  are integers. Then  $u^a \in I^{(han)}$  and  $I^{hn}u^a \subseteq I^{hr}u^a \subseteq I^{(han+hr)} = I^{(hq)}$ . We now come to a key point: we can show that

$$I^{(hq)} \subseteq I^{[q]}. \quad (*)$$

To see this, note that because the Frobenius endomorphism is flat for regular rings,  $I^{[q]}$  has no associated primes other than the minimal primes of  $I$ , and it suffices to check  $(*)$  after localizing at each minimal prime  $P$  of  $I$ . But after localization,  $I$  has at most  $h$  generators, and so each monomial of degree  $hq$  in these generators is a multiple of the  $q$ -th power of at least one of the generators. This completes the proof of  $(*)$ . Taking  $n$ -th powers gives that  $I^{hn^2}u^{an} \subseteq (I^{[q]})^n = (I^n)^{[q]}$ , and since  $q \geq an$ , we have  $I^{hn^2}u^q \subseteq (I^n)^{[q]}$  for fixed  $h$  and  $n$  and all  $q$ . Let  $d$  be any nonzero element of  $I^{hn^2}$ . The condition that  $du^q \in (I^n)^{[q]}$

for all  $q$  says precisely that  $u$  is in the tight closure of  $I^n$  in  $R$ . But in a regular ring, every ideal is tightly closed, and so  $u \in I^n$ , as required.

## 6. Extending the Theory to Affine Algebras in Characteristic 0

In this section we discuss briefly how to extend the results of tight closure theory to finitely generated algebras over a field  $K$  of characteristic zero. There is a good theory with essentially the same properties as in positive characteristic. See [Hochster and Huneke 1999; Hochster 1996].

Suppose that we have a finitely generated  $K$ -algebra  $R$ . We may think of  $R$  as having the form  $K[x_1, \dots, x_n]/(f_1, \dots, f_m)$  for finitely many polynomials  $f_j$ . An ideal  $I \subseteq R$  can be given by specifying finitely many polynomials  $g_j \in T = K[x_1, \dots, x_n]$  that generate it, and an element  $u$  of  $R$  can be specified by giving a polynomial  $h$  that maps to. We can then choose a finitely generated  $\mathbb{Z}$ -subalgebra  $B$  of  $K$  that contains all of the coefficients of the  $f_j$ , the  $g_j$  and of  $h$ . We can form a ring  $R_B = B[x_1, \dots, x_n]/(f_1, \dots, f_m)$  and we can consider the ideal  $I_B$  of the  $R_B$  generated by the images of the  $g_j$  in  $R_B$ . It turns out that after localizing  $B$  at one nonzero element we can make other pleasant assumptions: that  $I_B \subseteq R_B \subseteq R$ , that  $R_B$  and  $R_B/I_B$  are  $B$ -free (the lemma of generic freeness), and that tensoring with  $K$  over  $B$  converts  $I_B \subseteq R_B$  to  $I \subseteq R$ . Moreover,  $h$  has an image in  $R_B \subseteq R$  that we may identify with  $u$ .

We then define  $u$  to be in the tight closure of  $I$  in  $R$  provided that for all maximal  $Q$  in a dense open subset of the maximal spectrum of  $B$ , with  $\kappa = B/Q$ , the image of  $u$  in  $R_\kappa = \kappa \otimes_B R_B$  is in the characteristic- $p$  tight closure of  $I_\kappa = IR_\kappa$  — this makes sense because  $B/Q$  will be a finite field.

This definition turns out to be independent of the choices of  $B$ ,  $R_B$ ,  $I_B$ , etc.

Here is one very simple example. Let  $R = K[x, y, z]/(x^3 + y^3 + z^3)$  where  $K$  is any field of characteristic 0, e.g., the complex numbers, let  $I = (x, y)$  and  $u$  be the image of  $z^2$ . In this case we may take  $B = \mathbb{Z}$ ,  $R_\mathbb{Z} = \mathbb{Z}[x, y, z]/(x^3 + y^3 + z^3)$  and  $I_\mathbb{Z} = (x, y)R_\mathbb{Z}$ . Then  $z^2$  is in the characteristic 0 tight closure of  $(x, y)R$  because for every prime integer  $p \neq 3$  (these correspond to the maximal ideals of  $\mathbb{Z}$ , the image of  $z^2$  is in the characteristic- $p$  tight closure of  $(x, y)(\mathbb{Z}/p\mathbb{Z})[x, y, z]/(x^3 + y^3 + z^3)$ ). Take  $c = x$ , for example. One can check that  $c(z^2)^q \in (x^q, y^q)(R_\mathbb{Z}/pR_\mathbb{Z})$  for all  $q = p^e$ . Write  $2q = 3k+a$ ,  $a \in \{1, 2\}$ , and use that  $xz^2q = \pm x(x^3 + y^3)^k z^a$ . Each term in  $x(x^3 + y^3)^k$  has the form  $x^{3i+1}y^{3j}$  where  $3i+3j = 3k \geq 2q-2$ . Since  $(3i+1)+3j \geq 2q-1$ , at least one of the exponents is  $\geq q$ .

## 7. Test Elements

In this section we again study the case of rings of characteristic  $p > 0$ . Let  $R$  be a Noetherian domain. We shall say that an element  $c \in R - \{0\}$  is a *test element* if for every ideal  $I$  of  $R$ ,  $cI^* \subseteq I$ . An equivalent condition is that for

every ideal  $I$  and element  $u$  of  $R$ ,  $u \in I^*$  if and only if  $cu^q \in I^{[q]}$  for every  $q = p^e \geq 1$ . The reason that this holds is the easily verified fact that if  $u \in I^*$ , then  $u^q \subseteq (I^{[q]})^*$  for all  $q$ . Thus, an element that is known to be a test element can be used in all tight closure tests. A priori the element used in tight closure tests for whether  $u \in I^*$  in the definition of tight closure can vary with both  $I$  and  $u$ . The test elements together with 0 form an ideal called the *test ideal*.

Test elements are known to exist for domains finitely generated over a field. Any element  $d \neq 0$  such that  $R_d$  is regular turns out to have a power that is a test element. We won't prove this here.

We will explain, however, why the Jacobian ideal of a domain finitely generated over an algebraically closed field is contained in the test ideal, which is one of the ingredients of Theorem 1.4. The discussion of the results on test elements needed for Theorem 1.4 is continued in the next section.

Here is a useful result that leads to existence theorems for test elements.

**THEOREM 7.1.** *Let  $R$  be a Noetherian domain module-finite over a regular domain  $A$  of characteristic  $p > 0$ , and suppose that the extension of fraction fields is separable. Then:*

- (a) *There are elements  $d \in A - \{0\}$  such that  $dR^{1/p} \subseteq R[A^{1/p}]$ .*
- (b) *For any  $d$  as in part (a), the element  $c = d^2$  satisfies*

$$cR^{1/q} \subseteq R[A^{1/q}] \quad \text{for all } q. \tag{\dagger}$$

Let  $R_q = R[A^{1/q}]$ .

- (c) *Any element  $c \neq 0$  of  $R$  that satisfies condition  $(\dagger)$  is a test element for  $R$ .*

*Thus,  $R$  has test elements.*

**PROOF.** If we localize at all nonzero elements of  $A$  we are in the case where  $A$  is a field and  $R$  is a separable field extension. This is well-known and is left as an exercise for the reader. It follows that  $R^{1/p}/R[A^{1/p}]$ , which we may think of as a finitely generated  $A^{1/p}$ -module, is a torsion module. But then it is killed by an element of  $A^{1/p} - \{0\}$  and, hence, by an element of  $A - \{0\}$ .

For part (b) note we note that since  $dR^{1/p} \subseteq R[A^{1/p}]$ , we have  $d^{1/q}R^{1/pq} \subseteq R^{1/q}[A^{1/pq}]$  for all  $q = p^e$ . Thus,

$$d^{1+1/p}R^{1/p^2} \subseteq d(d^{1/p}R^{1/p^2}) \subseteq d(R^{1/p}[A^{1/p^2}]) \subseteq R[A^{1/p}][A^{1/p^2}] = R[A^{1/p^2}].$$

Continuing in this way, one concludes easily by induction that

$$d^{1+1/p+\dots+1/p^{e-1}}R^{1/p^e} \subseteq R[A^{1/p^e}].$$

Since  $2 > 1 + 1/p + \dots + 1/p^{e-1}$  for all  $p \geq 2$ , we obtain the desired result.

Finally, suppose that  $c$  satisfies condition  $(\dagger)$ . It suffices to show that for all  $I$  and  $u \in I^*$ , that  $cu \in I$ . But if  $u \in I^*$  we can choose  $a \in A - \{0\}$  (all nonzero elements of  $R$  have nonzero multiples in  $A$ ) such that  $au^q \in I^{[q]}$  for all  $q = p^e$ .

Taking  $q$ -th roots gives  $a^{1/q}u \in IR^{1/q}$  for all  $q$ . Multiplying by  $c$  gives that  $a^{1/q}cu \in IR[A^{1/q}] = IR_q$  for all  $q$ , and so  $a^{1/q} \in IR_q :_{R_q} cu$  for all  $q$ . It is not hard to show that  $R \otimes_A A^{1/q} \cong R[A^{1/q}]$  here. (The obvious map is onto, and since  $R$  is torsion-free over  $A$  and  $A^{1/q}$  is  $A$ -flat,  $R \otimes_A A^{1/q}$  is torsion-free over, so that we can check injectivity after localizing at  $A - \{0\}$ , and we thus reduce to the case where  $A$  is a field and  $R$  is a finite separable extension field, where the result is the well-known linear disjointness of separable and purely inseparable field extensions.) The flatness of Frobenius for  $A$  means precisely that  $A^{1/q}$  is flat over  $A$ , so that  $R_q$  is flat over  $R$ ; this is simply a base change. Thus,  $IR_q :_{R_q} cu = (I :_R cu)R_q \subseteq (I :_R cu)R^{1/q}$ . Hence, for all  $q = p^e$ ,  $a^{1/q} \in JR^{1/q}$ , where  $J = I :_R cu$ . This shows that  $a \in J^{[q]}$  for all  $q$ . Since  $a \neq 0$ , we must have that  $J$  is the unit ideal, i.e., that  $cu \in I$ .

The same argument works essentially without change when  $I$  is a submodule of a free module instead of an ideal.  $\square$

## 8. Test Elements Using the Lipman–Sathaye Theorem

This section describes material from [Hochster and Huneke 1999, Section 1.4].

For the moment, we do not make any assumption on the characteristic. Let  $T \subseteq R$  be a module-finite extension, where  $T$  is a Noetherian domain,  $R$  is torsion-free as a  $T$ -module and the extension is generically smooth. Thus, if  $\mathcal{K}$  is the fraction field of  $T$  and  $\mathcal{L} = \mathcal{K} \otimes_T R$  is the total quotient ring of  $R$  then  $\mathcal{K} \rightarrow \mathcal{L}$  is a finite product of separable field extensions of  $\mathcal{K}$ . The *Jacobian ideal*  $\mathcal{J}(R/T)$  is defined as the 0-th Fitting ideal of the  $R$ -module of Kähler  $R$ -differentials  $\Omega_{R/T}$ , and may be calculated as follows: write  $R \cong T[X_1, \dots, X_n]/P$  and then  $\mathcal{J}(R/T)$  is the ideal generated in  $R$  by the images of all the Jacobian determinants  $\partial(g_1, \dots, g_n)/\partial(X_1, \dots, X_n)$  for  $n$ -tuples  $g_1, \dots, g_n$  of elements of  $P$ . Moreover, to generate  $\mathcal{J}(R/T)$  it suffices to take all the  $n$ -tuples of  $g_i$  from a fixed set of generators of  $P$ .

Now suppose in addition that  $T$  is regular. Let  $R'$  be the integral closure of  $R$  in  $\mathcal{L}$ , which is well known to be module-finite over  $T$  (the usual way to argue is that any discriminant multiplies it into a finitely generated free  $T$ -module). Let  $J = \mathcal{J}(R/T)$  and  $J' = \mathcal{J}(R'/T)$ . The result of Lipman and Sathaye [1981, Theorem 2, p. 200] may be stated as follows:

**THEOREM 8.1 (LIPMAN–SATHAYE).** *With notation as above (in particular, there is no assumption about the characteristic, and  $T$  is regular), suppose also that  $R$  is an integral domain. If  $u \in \mathcal{L}$  is such that  $uJ' \subseteq R'$  then  $uJR' \subseteq R$ . In particular, we may take  $u = 1$ , and so  $JR' \subseteq R$ .*  $\square$

This property of “capturing the normalization” will enable us to produce test elements.

**COROLLARY 8.2 (EXISTENCE OF TEST ELEMENTS VIA THE LIPMAN–SATHAYE THEOREM).** *If  $R$  is a domain module-finite over a regular domain  $A$  of character-*

*istic  $p$  such that the extension of fraction fields is separable, then every element  $c$  of  $J = \mathcal{J}(R/A)$  is such that  $cR^{1/q} \subseteq A^{1/q}[R]$  for all  $q$ , and, in particular,  $cR^\infty \subseteq A^\infty[R]$ . Thus, if  $c \in J \cap (R - \{0\})$ , it is a test element.*

PROOF. Since  $A^{1/q}[R] \cong A^{1/q} \otimes_A R$ , the image of  $c$  is in  $\mathcal{J}(A^{1/q}[R]/A^{1/q})$ , and so the Lipman–Sathaye theorem implies that  $c$  multiplies the normalization  $S$  of  $A^{1/q}[R]$  into  $A^{1/q}[R]$ . Thus, it suffices to see that  $R^{1/q}$  is contained in  $S$ . Since it is clearly integral over  $A^{1/q}[R]$  (it is obviously integral over  $R$ ), we need only see that the elements of  $R^{1/q}$  are in the total quotient ring of  $A^{1/q}[R]$ , and for this purpose we may localize at  $A^\circ = A - \{0\}$ . Thus, we may replace  $A$  by its fraction field and assume that  $A$  is a field, and then  $R$  is replaced by  $(A^\circ)^{-1}R$ , which is a separable field extensions. Thus, we come down to the fact that if  $A \subseteq R$  is a finite separable field extension, then the injection  $A^{1/q} \otimes_A R \rightarrow R^{1/q}$  (the map is an injection because separable and purely inseparable field extensions are linearly disjoint) is an isomorphism, which is immediate by a degree argument.  $\square$

COROLLARY 8.3 (MORE TEST ELEMENTS VIA LIPMAN–SATHAYE). *Let  $K$  be a field of characteristic  $p$  and let  $R$  be a  $d$ -dimensional geometrically reduced (i.e., the ring stays reduced even when one tensors with an inseparable extension of  $K$  —this is automatic if  $K$  is perfect) domain over  $K$  that is finitely generated as a  $K$ -algebra. Let  $R = K[x_1, \dots, x_n]/(g_1, \dots, g_r)$  be a presentation of  $R$  as a homomorphic image of a polynomial ring. Then the  $(n-d) \times (n-d)$  minors of the Jacobian matrix  $(\partial g_i / \partial x_j)$  are contained in the test ideal of  $R$ , and remain so after localization and completion. Thus, any element of the Jacobian ideal generated by all these minors that is in  $R - \{0\}$  is a test element.*

PROOF. We pass to  $K(t) \otimes_K R$ , if necessary, where  $K(t)$  is a simple transcendental extension of  $K$ , to guarantee that the field is infinite. Our hypothesis remains the same, the Jacobian matrix does not change, and, since  $K(t) \otimes_K R$  is faithfully flat over  $R$ , it suffices to consider the latter ring. Thus, we may assume without loss of generality that  $K$  is infinite. The calculation of the Jacobian ideal is independent of the choice of indeterminates. We are therefore free to make a linear change of coordinates, which corresponds to choosing an element of  $G = GL(n, K) \subseteq K^{n^2}$  to act on the one-forms of  $K[x_1, \dots, x_n]$ . For a dense Zariski open set  $U$  of  $G \subseteq K^{n^2}$ , if we make a change of coordinates corresponding to an element  $\gamma \in U \subseteq G$  then, for every choice of  $d$  of the (new) indeterminates, if  $A$  denotes the  $K$ -subalgebra of  $R$  that these  $d$  new indeterminates generate, the two conditions listed below will hold:

- (1)  $R$  will be module-finite over  $A$  (and the  $d$  chosen indeterminates will then, perforce, be algebraically independent).
- (2)  $R$  will be generically smooth over  $A$ .

We may consider these two statements separately, for if each holds for a dense Zariski open subset of  $G$  we may intersect the two subsets. The first state-

ment follows from the standard “linear change of variable” proofs of the Noether normalization theorem for affine  $K$ -algebras (these may be used whenever the ring contains an infinite field). For the second, we want each  $d$  element subset, say, after renumbering,  $x_1, \dots, x_d$ , of the variables to be a separating transcendence basis for the fraction field  $L$  of  $R$  over  $K$ . (The fact that  $R$  is geometrically reduced over  $K$  implies that  $L$  is separably generated over  $K$ .) By [Kunz 1986, Theorem 5.10(d)], for example, a necessary and sufficient condition for  $x_1, \dots, x_d$  to be a separating transcendence basis is that the differentials of these elements  $dx_1, \dots, dx_d$  in  $\Omega_{L/K} \cong L^d$  be a basis for  $\Omega_{L/K}$  as an  $L$ -vector space. Since the differentials of the original variables span  $\Omega_{L/K}$  over  $L$ , it is clear that the set of elements of  $G$  for which all  $d$  element subsets of the new variables have differentials that span  $\Omega_{L/K}$  contains a Zariski dense open set.

Now suppose that a suitable change of coordinates has been made, and, as above, let  $A$  be the ring generated over  $K$  by some set of  $d$  of the elements  $x_i$ . The  $n-d$  size minors of  $(\partial g_i / \partial x_j)$  involving the  $n-d$  columns of  $(\partial g_i / \partial x_j)$  corresponding to variables not chosen as generators of  $A$  precisely generate  $\mathcal{J}(R/A)$ .  $R$  is module-finite over  $A$  by the general position argument, and since it is equidimensional and reduced, it is likewise torsion-free over  $A$ , which is a regular domain. It is generically smooth likewise, because of the general position of the variables. The result is now immediate from Corollary 8.2: as we vary the set of  $d$  variables, every  $n-d$  size minor occurs as a generator of some  $\mathcal{J}(R/A)$ .  $\square$

## 9. Tight Closure for Submodules

We make some brief remarks on how to extend the theory of tight closure to submodules of arbitrary modules.

Let  $R$  be a Noetherian ring of positive prime characteristic  $p$  and let  $G$  be a free  $R$ -module with a specified free basis  $u_j$ , which we allow to be infinite. Then we may define an action of the Frobenius endomorphism  $F$  and its iterates on  $G$  very simply as follows: if  $g = \sum_{i=1}^t r_i u_{j_i}$  (where the  $j_i$  are distinct) we let  $F^e(g)$ , which we also denote  $g^{p^e}$ , be  $\sum_{i=1}^t r_i^{p^e} u_{j_i}$ . Thus, we are simply letting  $F$  act (as it does on the ring) on all the coefficients that occur in the representation of an element of  $G$  in terms of the free basis. If  $N \subseteq G$  is a submodule, we let  $N^{[p^e]}$  denote the submodule of  $G$  spanned by all the elements  $g^{p^e}$  for  $g \in N$ . We then define an element  $x \in G$  to be in  $N^*$  if there exists  $c \in R^\circ$  such that  $cx^{p^e} \in N^{[p^e]}$  for all  $e \gg 0$ .

More generally, if  $M$  is any  $R$ -module,  $N$  is a submodule, and we want to determine whether  $x \in M$  is in the tight closure  $N^*$  of  $N$  in  $M$ , we can proceed by mapping a free module  $G$  onto  $M$ , taking an element  $g \in G$  that maps to  $x$ , letting  $H$  be the inverse image of  $N$  in  $G$ , and letting  $x$  be in  $N_M^*$  precisely when  $g \in H_G^*$ , where we are using subscripts to indicate the ambient module. This definition turns out to be independent of the choice of free module  $G$  mapping onto  $M$ , and of the choice of free basis for  $G$ .

I believe that there are many important questions about the behavior of tight closure for modules that are not finitely generated over the ring, especially for Artinian modules over local rings. See question 3 in the next section.

However, for the rest of this section we restrict attention to the case of finitely generated modules. The theory of test elements for tight closure of ideals extends without change to the generality of modules.

In order to prove the result of Theorem 1.4 one may make use of a version of the phantom acyclicity theorem. We first recall the result of [Buchsbaum and Eisenbud 1973] concerning when a finite free complex over a Noetherian ring  $R$  is acyclic. Suppose that the complex is

$$0 \rightarrow R^{b_n} \rightarrow \cdots \rightarrow R^{b_0} \rightarrow 0$$

and that  $r_i$  is the (determinantal) rank of the matrix  $\alpha_i$  giving the map from  $R^{b_i} \rightarrow R^{b_{i-1}}$ ,  $0 \leq i \leq n+1$ , where  $b_{n+1}$  is defined to be 0. The result of [Buchsbaum and Eisenbud 1973] is that the complex is acyclic if and only if

- (1) for  $0 \leq i \leq n$ ,  $b_i = r_{i+1} + r_i$ , and
- (2) for  $1 \leq i \leq n$ , the depth of the ideal  $J_i$  generated by the  $r_i$  size minors of  $\alpha_i$  is at least  $i$  (this is automatic if the ideal generated by the minors is the unit ideal; by convention, the unit ideal has depth  $+\infty$ ).

A complex  $0 \rightarrow G_n \rightarrow \cdots \rightarrow G_0 \rightarrow 0$  is said to be *phantom acyclic* if for all  $i \geq 1$ , one has that the kernel  $Z_i$  of  $G_i \rightarrow G_{i-1}$  is in the tight closure of the module of boundaries  $B_i$  (the image of  $G_{i+1}$  in  $G_i$ ) in  $G_i$ . Note that this implies that  $Z_i/B_i$  is killed by the test ideal.

Consider the following weakening of condition (2) above:

- (2°) for  $1 \leq i \leq n$ , the height of the ideal  $J_i$  generated by the  $r_i$  size minors of  $\alpha_i$  is at least  $i$  (this is automatic if the ideal generated by the minors is the unit ideal; by convention, the unit ideal has height  $+\infty$ ).

Then:

**THEOREM 9.1 (PHANTOM ACYCLICITY CRITERION).** *Let  $R$  be a reduced biequidimensional Noetherian ring of positive characteristic. A finite free complex as above is phantom acyclic provided that conditions (1) and (2°) hold.*

See [Hochster and Huneke 1990] and [Hochster and Huneke 1993] for detailed treatments where the result is established in much greater generality and a partial converse is proved, and to [Aberbach 1994] for the further development of the closely related notion of *finite phantom projective dimension*.

Note that in a domain, condition (2°) simply says that every  $J_i$  has height at least  $i$ : this replaces the subtle and difficult notion of “depth” by the much more tractable notion of “height” (or “codimension”).

Theorem 1.4 is simply the result of applying the phantom acyclicity criterion to a Koszul complex. Conditions (1) and (2) are easy to verify. Therefore, the higher homology is killed by the test ideal, which contains the Jacobian ideal.

There is another point of view that is very helpful in understanding the phantom acyclicity theorem. It involves the main result of [Hochster and Huneke 1991a]. If  $R$  is a domain, let  $R^+$  denote the integral closure of  $R$  in an algebraic closure of its fraction field, which is a maximal integral extension of  $R$  that is a domain. It is unique up to non-unique isomorphism. The theorem of [Hochster and Huneke 1991a] is that every system of parameters of  $R$  is a regular sequence in  $R^+$ : thus,  $R^+$  is a big Cohen–Macaulay algebra for  $R$  (and for any module-finite extension domain of  $R$ , all of which are embeddable in  $R^+$ ). Suppose that one has a complex that satisfies the hypothesis of the phantom acyclicity criterion. When one tensors with  $R^+$  it actually becomes acyclic: heights become depths in  $R^+$ , and one may apply a generalization to the non-Noetherian case of the acyclicity criterion of [Buchsbaum and Eisenbud 1973] presented in great detail in [Northcott 1976]. One may use this to see that any cycle becomes a boundary after tensoring with a sufficiently large but module-finite extension of  $R$ . The fact that the cycles are in the tight closure of the boundaries is now analogous to the fact that when an ideal  $I \subseteq R$  is expanded and then contracted from a module-finite extension  $S$  of  $R$ , we have  $IS \cap R \subseteq I^*$ : compare Theorem 3.6.

Finally, we mention the vanishing theorem for maps of Tor. Let  $A \subseteq R \rightarrow S$  be maps of rings of characteristic  $p$ , where  $A$  is regular,  $R$  is module-finite and torsion-free over  $A$ , and  $S$  is any regular ring. The map  $R \rightarrow S$  is arbitrary here: it need not be injective nor surjective. Let  $M$  be any  $R$ -module.

**THEOREM 9.2 (VANISHING THEOREM FOR MAPS OF TOR).** *With assumptions as just above, the maps  $\text{Tor}_i^A(M, R) \rightarrow \text{Tor}_i^A(M, S)$  are 0 for all  $i \geq 1$ .*

**SKETCH OF PROOF.** One may easily reduce to the case where  $S$  is complete local and then to the case where  $A$  is complete local. By a direct limit argument one may reduce to the case where  $M$  is finitely generated over  $A$ . Then  $M$  has a finite free resolution over  $A$ , which satisfies the hypothesis of the characterization of acyclic complexes given in [Buchsbaum and Eisenbud 1973]. When we tensor with  $R$  over  $A$  we get a free complex over  $R$  that satisfies the phantom acyclicity theorem: every cycle is in the tight closure of the boundaries. Taking its homology gives the  $\text{Tor}_i^A(M, R)$ . Now when we tensor  $S$ , every module is tightly closed, so the cycles coming from the complex over  $R$  are now boundaries, which gives the desired result.  $\square$

See [Hochster and Huneke 1990; 1993], the discussion in [Hochster and Huneke 1995], and [Ranganathan 2000]. This is an open question in mixed characteristic. This vanishing result is amazingly powerful. In the case where  $S$  is simply a field, it implies the direct summand conjecture, i.e., that regular rings are direct

summands of their module-finite extensions. In the case where  $S$  is regular and  $R$  is a direct summand of  $S$  it implies that  $R$  is Cohen–Macaulay. Both questions are open in mixed characteristic. The details of these implications are given in [Hochster and Huneke 1995]. In [Ranganathan 2000], it is shown, somewhat surprisingly, that the vanishing theorem for maps of Tor is actually equivalent to the following question about splitting: let  $R$  be a regular local ring, let  $S$  be a module-finite extension, and suppose that  $P$  is a height one prime ideal of  $S$  that contracts to  $xR$ , where  $x$  is a regular parameter in  $R$ . Then  $xR$  is a direct summand of  $S$  as an  $R$ -module.

## 10. Further Thoughts and Questions

What we have said about tight closure so far is only the tip of an iceberg. Here are some major open questions.

1. Does tight closure commute with localization under mild assumptions on the ring? This is not known to be true even for finitely generated algebras over a field. Aspects of the problem are discussed in [Aberbach et al. 1993; Hochster and Huneke 2000; Vraciu 2000].
2. Under mild conditions, if a ring has the property that every ideal is tightly closed, does that continue to hold when one localizes? This is not known for finitely generated algebras over a field, nor for complete local rings. An affirmative answer to 1. would imply an affirmative answer to 2.

Rings such that every ideal is tightly closed are called *weakly F-regular*. The word “weakly” is omitted if this property also holds for all localizations of the ring. Weakly F-regular rings are Cohen–Macaulay and normal under very mild conditions—this holds even if one only assumes that ideals generated by parameters are tightly closed (this weaker property is called *F-rationality* and is closely related to the notion of rational singularities; see [Hara 1998; Smith 1997; Vélez 1995; Enescu 2000]). Both of the conditions of weak F-regularity and F-rationality tend to imply that the singularities of the ring are in some sense good. However, the theory is complicated [Hara and Watanabe 2002]. It is worth noting that weak F-regularity does not deform [Singh 1999], and that direct summands of F-rational rings are not necessarily F-rational [Watanabe 1997]. See also [Hara et al. 2002a; 2002b]. Weak F-regularity is established for some important classes of rings (those defined by the vanishing of the minors of fixed size of a matrix of indeterminates, and homogeneous coordinate rings of Grassmannians) in [Hochster and Huneke 1994b, Theorem 7.14].

3. Let  $M$  be an Artinian module over, say, a complete reduced local ring with a perfect residue field. Let  $N$  be a submodule of  $M$ . Is it true that  $u \in N_M^*$  if and only if there exists  $Q$  with  $N \subseteq Q \subseteq M$  with  $Q/N$  of finite length such that  $u \in N_Q^*$ ? This is true in a graded version and for isolated singularities [Lyubeznik and Smith 1999; 2001]; other cases are established in [Elitzur 2003].

For any domain  $R$ , let  $R^+$  denote the integral closure of  $R$  in an algebraic closure of its fraction field. This is unique up to non-unique isomorphism, and may be thought of as a “largest” domain extension of  $R$  that is integral over  $R$ .

**4.** For an excellent local domain  $R$ , is an element  $r \in R$  in the tight closure of  $I$  if and only if it is in  $IS$  for some module-finite extension domain of  $R$ ? It is equivalent to assert that for such a local domain  $R$ ,  $I^* = IR^+ \cap R$ . This is known for ideals generated by part of a system of parameters [Smith 1994]. It is known that  $IR^+ \cap R \subseteq I^*$ . For some results on homogeneous coordinate rings of elliptic curves, see the remarks following the next question.

It is known in characteristic  $p$  that for a complete local domain  $R$ , and element  $u \in R$  is in  $I^*$  if and only if it is in  $IB \cap R$  for some big Cohen–Macaulay algebra extension ring  $B$  of  $R$ : see [Hochster 1994a, Section 11].

It is worth mentioning that there is an intimate connection between tight closure and the existence of big Cohen–Macaulay algebras  $B$  over local rings  $(R, m)$ , i.e., algebras  $B$  such that  $mB \neq B$  and every system of parameters for  $R$  is a regular sequence on  $B$ . Tight closure ideas led to the proof in [Hochster and Huneke 1992] that if  $R$  is an excellent local domain of characteristic  $p$  then  $R^+$  is a big Cohen–Macaulay algebra. Moreover, for complete local rings  $R$ , it is known [Hochster 1994a, Section 11] that  $u \in I^*$  if and only if  $R$  has a big Cohen–Macaulay algebra  $B$  such that  $u \in IB$ .

**5.** Is there an effective way to compute tight closures? The answer is not known even for ideals of cubical cones, i.e., of rings of the form  $K[X, Y, Z]/(X^3 + Y^3 + Z^3)$  in positive characteristic different from 3. However, in cones over elliptic curves, tight closure agrees with plus closure (i.e., with  $IR^+ \cap R$ ) for *homogeneous ideals  $I$  primary to the homogeneous maximal ideal*: see [Brenner 2003b; 2002]. For ideals that are not homogeneous, the question raised in 4. is open even for such rings. When the characteristic of  $K$  is congruent to 2 mod 3, it is even possible that tight closure agrees with Frobenius closure in these rings. See [McDermott 2000; Vraciu 2002].

**6.** How can one extend tight closure to mixed characteristic? By far the most intriguing result along these lines is due to Ray Heitmann [2002], who has proved that if  $(R, m)$  is a complete local domain of dimension 3 and mixed characteristic  $p$ , then every Koszul relation on parameters in  $R^+$  is annihilated by multiplication by arbitrarily small positive rational powers of  $p$  (that is, by  $p^{1/N}$  for arbitrarily large integers  $N$ ). This implies that regular local rings of dimension 3 are direct summands of their module-finite extension rings. Heitmann’s result can be used to prove the existence of big Cohen–Macaulay algebras in dimension 3: see [Hochster 2002]. Other possibilities are explored in [Hochster 2003] and [Hochster and Vélez 2004].

## Appendix: Some Examples in Tight Closure

BY GRAHAM J. LEUSCHKE

Tight closure and related methods in the study of rings of prime characteristic have taken on central importance in commutative algebra, leading to both new results and improvements on old ones. Unfortunately, tight closure has a reputation for inaccessibility to novices, with what can seem a bewildering array of F- prefixes and other terminology. The very definition of tight closure is less than immediately illuminating:

**DEFINITION A.1.** Let  $R$  be a Noetherian ring of prime characteristic  $p$ . Let  $I$  be an ideal of  $R$ . An element  $x \in R$  is said to be in the *tight closure* of  $I$  if there exists an element  $c$ , not in any minimal prime of  $R$ , so that for all large enough  $q = p^e$ ,  $cx^q \in I^{[q]}$ , where  $I^{[q]}$  is the ideal generated by the  $q$ -th powers of the elements of  $I$ . In this case we write  $x \in I^*$ .

This appendix is based on an hour-long help session about tight closure that I gave at MSRI following the series of lectures by Mel Hochster that constitute the bulk of this article. The help session itself was quite informal, driven mostly by questions from the audience, with the goal of presenting enough examples of computations to give a feeling for how the definition is used. The reader will quickly see that the methods are largely *ad hoc*; in fact, at this time there is no useful algorithm for determining that a given element is or is not in a certain tight closure.<sup>2</sup> Still, certain patterns will arise that indicate how problems of this sort are generally solved.

We first discuss the examples. Examples A.1 and A.2 are drawn from [Huneke 1998]. Example A.3 was shown me by Moira McDermott, whom I thank here for her help and insight into some of these computations.

After that, I present a few auxiliary results on tight closure, including the Strong Vanishing Theorem for hypersurfaces and some material on test elements. This section serves several purposes: In addition to putting the examples in context, the results address some of the audience questions raised during the help session and make this appendix relatively self-contained. I am grateful to Sean Sather-Wagstaff for his notes from the help session on this material.

Throughout, we work with Noetherian rings containing a field  $k$  of positive characteristic  $p$ , and write  $q$  for a varying power of  $p$ . Variables will be represented by capital letters, which we routinely decapitalize to indicate their images in a quotient ring.

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<sup>2</sup>See, however, [Sullivant 2002] for a procedure for calculating tight closures of monomial ideals in Fermat rings. Also, there is an algorithm due to Hochster for countable affine rings which involves enumerating all module-finite algebras over the ring. It is effective whenever tight closure is known to be the same as *plus closure*, but is impractical to implement.

**The examples.** We begin with the canonical first example of tight closure. It involves the “cubical cone” or “Fermat cubic” ring

$$R = k[\![X, Y, Z]\!]/(X^3 + Y^3 + Z^3),$$

which is in some sense the first nontrivial ring from the point of view of tight closure.

EXAMPLE A.1. Let  $R = k[\![X, Y, Z]\!]/(X^3 + Y^3 + Z^3)$ , where  $k$  is a field of characteristic  $p \neq 3$ , and let  $I = (y, z)$ . Then  $I^* = (x^2, y, z)$ .

We know from the example at the end of Section 6 (page 192) that  $x^2 \in (y, z)^*$ ; we will reproduce the argument here, since it has a flavor to which we should become accustomed. We will take  $c = z$  in the definition of tight closure, so we will show that  $z(x^2)^q \in (y, z)^{[q]} = (y^q, z^q)$  for all  $q = p^e$ . For a general  $q$ , write  $2q = 3u + i$ , where  $i$  is 1 or 2. Expand  $z(x^2)^q$ :

$$\begin{aligned} z(x^2)^q &= zx^{3u+i} = zx^i(x^3)^u = (-1)^u zx^i(y^3 + z^3)^u \\ &= (-1)^u zx^i \sum_{j=0}^u \binom{u}{j} y^{3j} z^{3(u-j)}. \end{aligned}$$

Consider a monomial  $x^i y^{3j} z^{3(u-j)+1}$  in this sum. If we have both  $3j \leq q-1$  and  $3(u-j)+1 \leq q-1$ , then  $3u+1 \leq 2q-2$ , so  $2q \geq 3u+3$ , a contradiction. Therefore each monomial in the expansion of  $z(x^2)^q$  has degree at least  $q$  in either  $y$  or  $z$ , that is, each monomial is in  $(y^q, z^q)$ , as desired.

Now we need only show that  $x \notin I^*$ . This argument is due to Mordechai Katzman, by way of [Huneke 1998]. We take for granted that  $z^N$  is a test element for some large  $N$ , that is,  $z^N$  can be used as  $c$  in any and all tight closure tests (see Definition A.2 and Theorem A.9). Then  $x \in I^*$  if and only if  $z^N x^q \in (y^q, z^q)$  for all  $q$ . Choose  $q$  to be larger than  $N$  and let  $J = (X^3 + Y^3 + Z^3, Y^q, Z^q) \subseteq k[X, Y, Z]$ .

Let  $>$  be the reverse lexicographic term order on  $k[X, Y, Z]$  with  $X > Y > Z$ . Then the initial ideal  $\text{in}_{>}(J)$  is  $(X^3, Y^q, Z^q)$ . Write  $q = 3u + i$ , where  $i$  is either 1 or 2. Then  $X^q = X^{3u+i} \equiv (-1)^u (Y^3 + Z^3)^u X^i$  modulo  $J$ . We also have

$$\text{in}_{>}(Z^N X^q) = \text{in}_{>}(Z^N (-1)^u (Y^3 + Z^3)^u X^i) = X^i Y^{3u} Z^N.$$

Since  $N < q$ , this last is not in  $\text{in}_{>}(J) = (X^3, Y^q, Z^q)$ , and we see that  $Z^N X^q$  is not in  $J$ . Thus  $z^N x^q \notin (y^q, z^q)$  in  $k[X, Y, Z]/(X^3 + Y^3 + Z^3)$ . The same holds in  $R$  since  $k[X, Y, Z]/J$  has finite length.

At this point in the help session, an audience member asked, “What difference do the numbers make?” That is, are the exponents (3, 3, 3 in the case of the Fermat cubic) vital to the outcome of the example? The next example, a side-by-side comparison, shows that they are indeed.

EXAMPLE A.2. Let  $S = k[\![X, Y, Z]\!]$ , where the characteristic of  $k$  is greater than 7, and define two polynomials:  $f_1 = X^2 + Y^3 + Z^5$ , and  $f_2 = X^2 + Y^3 + Z^7$ . Let

$R_1 = S/(f_1)$  and  $R_2 = S/(f_2)$ , and put  $I_i = (y, z)R_i$  for  $i = 1, 2$ . Then  $I_1^* = I_1$ , whereas  $I_2^* = (x, y, z)R_2$ .

We take for granted that for some large  $N$ ,  $x^N$  is a test element for ideals of both rings, that is, we may take  $c = x^N$  in the definition of tight closure (see Theorem A.9).

For the first assertion, it suffices to show that  $x^N x^q \notin (y^q, z^q)$  for some  $q = p^e$ . Since  $p > 5$ ,  $p$  is relatively prime to 30, and after possibly increasing  $N$  slightly, we can find a power  $q$  of  $p$  so that  $q = 30u - N + 2$  for some  $u$ . Expand  $x^N x^q$ :

$$x^{N+q} = x^{30u+2} = (x^2)^{15u+1} = \pm(y^3 + z^5)^{15u+1} = \pm \sum_{j=0}^{15u+1} y^{3j} z^{5(15u-j+1)}.$$

To show that  $x^{q+1} \notin (y^q, z^q)$ , we just need to find  $j$  such that  $3j < q$  and  $5(15u - j + 1) < q$ . Taking  $j = 10u$  fills the bill. It remains only to show that the coefficient of  $y^{3(10u)} z^{5(5u+1)}$  is nonzero modulo  $p$ , that is, that the binomial coefficient  $\binom{15u+1}{10u}$  is not divisible by  $p$ . We must show that if a power of  $p$  divides the numerator of the fraction giving  $\binom{15u+1}{10u}$ , then it also divides the denominator. So suppose that  $p^a$  divides  $15u+1-j$  for some  $j \leq 5k$ . Then  $2p^a$  divides  $30u+2-j$ . Since  $q = p^e = 30u+1$ , we see that  $2p^a$  divides  $p^e - (2j-1)$ . It follows that  $p^a$  divides  $2j-1$ , which is a factor of the denominator, and we are done.

To see that  $I_2^* = (x, y, z)R_2$ , fix  $q$  and write  $N+q = 2u$ , again after increasing  $N$  if necessary. Then  $x^N x^q = x^{2u} = (-1)^u (y^3 + z^7)^u$ . Each monomial in the binomial expansion of the right-hand side is of the form  $y^{3j} z^{7(u-j)}$ . If both  $3j < q$  and  $7(u-j) < q$  for some  $j$ , then  $21j + 21(u-j) < 7q + 3q = 10q$ , forcing  $21u < 10q$ , or  $21u < 20u - 10$ , which is absurd. Thus, for each  $j$ , either  $3j \geq q$  or  $7(u-j) \geq q$ , which implies  $x^N x^q \in (y^q, z^q)$ , so  $x \in (y, z)^*$ .

In fact,  $R_1$  is *weakly F-regular*, which means that every ideal is tightly closed. On the other hand, we have shown above that  $R_2$  is not weakly F-regular.

The next example is due to M. McDermott. In addition to showing that the Strong Vanishing Theorem for hypersurfaces (Theorem A.8) is sharp, it illustrates the occasionally mysterious nature of tight closure computations: Sometimes the numbers just work out, especially when  $p$  is small.

**EXAMPLE A.3.** Let  $R = k[A, B, C, D, E]/(A^4 + B^4 + C^4 + D^4 + E^4)$ , where  $k$  is a field of characteristic  $p$ , and let  $I = (a^4, b^4, c^4, d^4)$ . By the Strong Vanishing Theorem for hypersurfaces, for  $p > 8$  we have  $I^* = I + R_{\geq 16} = I$  is tightly closed. For smaller  $p$ , though,  $I$  need not be tightly closed. In particular, when  $p = 7$  we have  $a^3 b^3 c^3 d^3 e^3 \in I^*$ .

To see this, let  $w = a^3 b^3 c^3 d^3 e^3$ . Then

$$w^7 = a^{21} b^{21} c^{21} d^{21} e^{21} = -a^{21} b^{21} c^{21} d^{21} e (a^4 + b^4 + c^4 + d^4)^5.$$

Every monomial of  $(a^4 + b^4 + c^4 + d^4)^5$  has at least one variable to the eighth power, so  $w^7 \in (a^{29}, b^{29}, c^{29}, d^{29}) \subseteq I^{[7]}$ . So in fact  $w^p \in I^{[p]}$  and  $w$  is in the

Frobenius closure of  $I$ . In particular, taking  $c = 1$  shows that  $w$  is in the tight closure of  $I$ .

The next example is the most involved we will consider. It is due originally to Anurag Singh [1998], though the proof we will present is due to Holger Brenner, with improvements by Singh and Huneke. It returns to the Fermat cubic of Example A.1.

**EXAMPLE A.4.** Let  $R = k[[X, Y, Z]]/(X^3 + Y^3 + Z^3)$ , where  $k$  is a field of characteristic  $p > 3$ . Then  $xyz \in (x^2, y^2, z^2)^*$ .

We will need the following lemma, which is an easy consequence of colon-capturing [Huneke 1998, Theorem 2.3].

**LEMMA A.5.** *Let  $R$  be a complete equidimensional local ring with a test element  $c$ . Let  $x_1, \dots, x_n, y$  be part of a system of parameters, and set  $I = (x_1, \dots, x_n)$ . Then for any ideal  $J$  and any element  $h \in R$ ,  $hy \in (I + yJ)^*$  if and only if  $h \in (I + J)^*$ .*

**PROOF.** Assume first that  $hy \in (I + yJ)^*$ . This happens if and only if for every  $q$  we have  $c(yh)^q \in (I + yJ)^{[q]}$ , which is equal to  $I^{[q]} + y^q J^{[q]}$ . So this happens if and only if there exists some  $a_q \in J^{[q]}$  such that  $y^q(ch^q - a_q) \in I^{[q]}$ . Now, by colon-capturing [Huneke 1998, Theorem 2.3],  $I^{[q]} : y^q \subseteq (I^{[q]})^*$ , so  $c(ch^q - a_q) \in I^{[q]}$ . Unraveling this one more time gives  $c^2 h^q \in I^{[q]} + J^{[q]}$ , as desired. The converse follows by retracing these steps.  $\square$

Returning to the example, we see by the Lemma that the claim is equivalent to showing that  $xy^2z \in (x^2, y^3, yz^2)^* = (x^2, z^3, yz^2)^*$ . This in turn is equivalent to showing that  $xy^2 \in (x^2, z^2, yz)^*$ .

For  $q = p^e$ , write  $q = 3u + i$ , where  $i$  is 1 or 2. We will take  $c = x^{3-i}y^{6-2i}$  in the definition of tight closure. First, expand:

$$\begin{aligned} y^{6-2i}y^{2q} &= y^{6-2i}y^{6u+2i} = y^3y^{3(2u+1)} = y^3(-1)^{2u+1}(x^3+z^3)^{2u+1} \\ &= y^3(-1)^{2u+1} \sum_{j=0}^{2u+1} \binom{2u+1}{j} x^{3(2u+1-j)} z^{3j}. \end{aligned}$$

We separate this sum into one part with a factor of  $x^q$  and one with a factor of  $z^q$ , writing the preceding expression as  $y^3(-1)^{2u+1}$  times  $Q$ , where

$$\begin{aligned} Q &= \left( \sum_{j=0}^u \binom{2u+1}{j} x^{3(2u+1-j)} z^{3j} + \sum_{j=u+1}^{2u+1} \binom{2u+1}{j} x^{3(2u+1-j)} z^{3j} \right) \\ &= \left( x^{3u+3} \sum_{j=0}^u \binom{2u+1}{j} x^{3(u-j)} z^{3j} + z^{3u+3} \sum_{j=u+1}^{2u+1} \binom{2u+1}{j} x^{3(2u+1-j)} z^{3(j-u-1)} \right) \\ &= \left( x^q x^{3-i} \sum_{j=0}^u \binom{2u+1}{j} x^{3(u-j)} z^{3j} + z^q z^{3-i} \sum_{j=0}^u \binom{2u+1}{j+u+1} x^{3(u-j)} z^{3j} \right). \end{aligned}$$

Thus  $y^{6-2i}y^{2q} = Fx^q + Gy^3z^q$ , where

$$\begin{aligned} F &= y^3(-1)^{2u+1}x^{3-i} \sum_{j=0}^u \binom{2u+1}{j} x^{3(u-j)} z^{3j}, \\ G &= (-1)^{2u+1}z^{3-i} \sum_{j=0}^u \binom{2u+1}{j+u+1} x^{3(u-j)} z^{3j}. \end{aligned}$$

We are trying to show that the element

$$(x^{3-i}y^{6-2i})x^qy^{2q} = x^{3-i}Fx^{2q} + Gx^{3-i+q}y^3z^q$$

is in the ideal  $(x^2, z^2, yz)^{[q]}$ . The term involving  $Fx^{2q}$  is taken care of, so it suffices to show that  $Gx^{3-i+q} \in (y, z)^{[q]}$ . Write  $Gx^{3-i+q}$  solely in terms of  $y$  and  $z$  (recall that  $q = 3u + i$ ):

$$\begin{aligned} Gx^{3-i+q} &= x^{3+3u}(-1)^{2u+1}z^{3-i} \sum_{j=0}^u \binom{2u+1}{j+u+1} x^{3(u-j)} z^{3j} \\ &= (-1)^{3u+2}(y^3+z^3)^{u+1}z^{3-i} \sum_{j=0}^u \binom{2u+1}{j+u+1} (-1)^{u-j}(y^3+z^3)^{u-j} z^{3j}. \end{aligned}$$

Each monomial in this sum has degree  $3(u+1) + (3-i) + 3u = 6u + 6 - i \geq 6u + 4 \geq 2q$ . Since the sum involves only  $y$  and  $z$ , each monomial must have degree at least  $q$  in either  $y$  or  $z$ , as desired.

One might reasonably ask why we chose to show the equivalent statement that  $xy^2 \in (x^2, z^2, yz)^*$ , rather than the originally claimed inclusion. The glib answer is that it works. A more considered and satisfying reply might point to the fact that we reduced the problem in the end to showing that  $Gx^{3-i+q} \in (y, z)^{[q]}$ , which was quite easy, and in the original formulation there was simply *too much* symmetry to make a similar reduction.

Brenner has recently used powerful geometric methods ([Brenner 2004] and [Brenner 2003a]) to prove results like the following, which vastly generalizes the example above.

**THEOREM A.6** [Brenner 2004, Corollary 9.3]. *Let  $k$  denote an algebraically closed field of characteristic 0 and let  $F \in k[x, y, z]$  denote a homogeneous polynomial of degree  $\delta$  such that  $R = k[x, y, z]/(F)$  is a normal domain. Let  $f_1, f_2, f_3 \in R$  denote  $R_+$ -primary homogeneous elements of degree  $d_1, d_2, d_3$ . Suppose that the sheaf of relations  $\mathcal{R}$  is indecomposable on the curve  $Y = \text{Proj } R$ . Then:*

- (i)  $R_m \subseteq (f_1, f_2, f_3)^*$  for  $m \geq \frac{d_1+d_2+d_3}{2} + \frac{\delta-3}{2}$ .
- (ii) For  $m < \frac{d_1+d_2+d_3}{2} - \frac{\delta+3}{2}$  we have  $(f_1, f_2, f_3)^* \cap R_m = (f_1, f_2, f_3) \cap R_m$ .

This follows from a more general fact involving semistability of vector bundles:

**THEOREM A.7** [Brenner 2004, Theorem 8.1]. *Let  $k$  denote an algebraically closed field of characteristic 0 and let  $R$  be a normal two-dimensional standard-graded  $k$ -algebra. Set  $t = \lceil \frac{d_1+\dots+d_n}{n-1} \rceil$ . Suppose that the sheaf of relations  $\mathcal{R}(m)$  for ideal generators  $f_1, \dots, f_n$  is semistable. Then*

$$(f_1, \dots, f_n)^* = (f_1, \dots, f_n) + R_{\geq t}.$$

**Auxiliary results.** We now mention a few results that were prepared for the help session, but were not presented. They are included here both as examples of “what might have been” and to make this appendix relatively self-contained (more so than the help session on which it is based). Complete proofs are given in [Huneke 1998].

The first result is the Strong Vanishing Theorem for hypersurfaces, which was mentioned in Example A.3.

**THEOREM A.8.** *Let  $R = k[X_0, \dots, X_d]/(f)$  be a quasi-homogeneous graded hypersurface over a field  $k$  of characteristic  $p > 0$ . Assume that  $R$  is an isolated singularity, and that the partials  $\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_d}$  form a system of parameters for  $R$ . If  $p > (d-1)(\deg f) - \sum_{i=1}^d \deg X_i$ , then for parameters  $y_1, \dots, y_d$  of degrees  $a_1, \dots, a_d$ ,*

$$(y_1, \dots, y_d)^* = (y_1, \dots, y_d) + R_{\geq a_1 + \dots + a_d}.$$

This theorem is particularly well-suited for computations; see [Sullivant 2002].

The audience at MSRI was interested in the theory of *test elements*, specifically when they are known to exist.

**DEFINITION A.2.** An element  $c$  of  $R$ , not in any minimal prime, is called a *test element* for ideals of  $R$  if  $xI^* \subseteq I$  for every ideal  $I \subseteq R$ . Equivalently,  $c$  can be used for all tight closure tests:  $x \in R$  is in  $I^*$  if and only if  $cx^q \in I^{[q]}$  for all  $q = p^e$ .

The most obvious immediate benefit of the existence of test elements is in showing that elements are *not* in tight closures. If  $c$  is known to be a test element, and it can be shown that  $cx^q \notin I^{[q]}$  for any one  $q$ , then  $x \notin I^*$ . We saw this principle in action in Example A.2.

The theorem below is not the most general result on the existence of test elements, but suffices for many applications. We say that a ring  $R$  of characteristic  $p$  is *F-finite* provided the ring of  $p$ -th roots  $R^{1/p}$  is a finitely generated  $R$ -module. A complete local ring  $(R, \mathfrak{m}, k)$  such that  $[k : k^p] < \infty$  is always F-finite, but there are many examples of rings, even fields, that are not.

**THEOREM A.9** [Hochster and Huneke 1994a, Prop. 6.23]. *Let  $R$  be reduced and F-finite or reduced and essentially of finite type over an excellent local ring. Let  $c$  be an element of  $R$  not in any minimal prime. If the localization  $R_c$  is Gorenstein and weakly F-regular, then  $c$  has a power which is a test element. In particular, if  $R_c$  is regular, then  $c$  has a power which is a test element.*

The second case of Theorem A.9, in which  $R$  is assumed to be essentially of finite type over an excellent local ring, is deduced from the F-finite case by means of the “ $\Gamma$ -construction” [Hochster and Huneke 1994a], which shows that such a ring has a faithfully flat extension  $R^\Gamma$  which is F-finite, and such that  $R_c^\Gamma$  is still weakly F-regular and Gorenstein. Then some power of  $c$  is a test element in  $R^\Gamma$ , and that property descends automatically from faithfully flat extensions.

Other, similar, sources of abundant test elements are the theorem of Lipman–Sathaye and its consequences (see Sections 7 and 8 above).

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MELVIN HOCHSTER  
 MATHEMATICS DEPARTMENT  
 UNIVERSITY OF MICHIGAN  
 EAST HALL  
 525 EAST UNIVERSITY  
 ANN ARBOR, MI 48109-1109  
 UNITED STATES  
 hochster@umich.edu

GRAHAM J. LEUSCHKE  
 DEPARTMENT OF MATHEMATICS  
 UNIVERSITY OF KANSAS  
 LAWRENCE, KS 66045  
 UNITED STATES  
 gjleusch@syr.edu, <http://www.leuschke.org/>