J. Pure and Appl. Algebra, (), 1–14 (2001)

Gorenstein modules, finite index, and finite Cohen–Macaulay type

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Abstract

A Gorenstein module over a local ring R is a maximal Cohen–Macaulay module of finite injective dimension. We use existence of Gorenstein modules to extend a result due to S. Ding: A Cohen–Macaulay ring of finite index, with a Gorenstein module, is Gorenstein on the punctured spectrum. We use this to show that a Cohen–Macaulay local ring of finite Cohen–Macaulay type is Gorenstein on the punctured spectrum. Finally, we show that for a large class of rings (including all excellent rings), the Gorenstein locus of a finitely generated module is an open set in the Zariski topology.

Key Words: Gorenstein module, canonical module, index, finite Cohen–Macaulay type

Introduction

Let $(R, \mathfrak{m})$ be a (commutative Noetherian) Cohen–Macaulay local ring. Various results in commutative algebra require that R have a *canonical module*, a maximal Cohen–Macaulay module of finite injective dimension and type one. Many of these remain true under the weaker hypothesis that R have a *Gorenstein module*, which may have type greater than one (Definition 1.1). The first example of a ring having a Gorenstein module but no canonical module appeared in [1], and [2] contains examples of Cohen–Macaulay local rings R_n, for each $n \geq 2$, having an indecomposable Gorenstein module of type n and no canonical module. J.-I. Nishimura [3] has given an example of an excellent Cohen–Macaulay UFD with a Gorenstein module, which has no canonical module. Hence it seems worthwhile to extend results whose proofs use a canonical module to this setting.

The aim of this paper is to use the existence of Gorenstein modules to derive information about the structure of R. In [4], S. Ding showed that if R has a canonical module, then R has finite index if and only if R is Gorenstein on the punctured spectrum (that is, $R_{\mathfrak{p}}$ is Gorenstein for all nonmaximal primes $\mathfrak{p}$ of R). A Cohen–Macaulay local ring $(R, \mathfrak{m})$ is said to have *finite index* provided there exists a positive integer n such that $R/\mathfrak{m}^n$ is not a homomorphic image of a maximal Cohen–Macaulay module with no nonzero free summand (see Section 2). Here we generalize Ding's results to a ring R having only a Gorenstein module. We also make connections, as in [5], between R being Gorenstein in codimension $k-1$ and modules satisfying (S_k) being k^{th} syzygies.

Section 3 gives applications of the preceding results to rings of finite Cohen–Macaulay type. We say that the Cohen–Macaulay local ring R has *finite Cohen–Macaulay type*, or *finite CM type*, provided R has, up to isomorphism, only finitely many indecomposable maximal Cohen–Macaulay modules. A CM local ring of finite CM type necessarily has finite index (see the proof of Theorem 3.3). Hence, under the additional hypothesis that R have a Gorenstein module, R is Gorenstein on the punctured spectrum. R. Wiegand and I showed in [6] that an excellent CM local ring of finite CM type is in fact regular on the punctured spectrum, but in light of Nishimura's example mentioned above, the two results are irredundant.

Throughout, all rings will be commutative rings with identity. We abbreviate by CM and MCM, respectively, the descriptors Cohen–Macaulay and maximal Cohen–Macaulay. Recall that a finitely generated nonzero R-module M is MCM provided the depth of M is equal to the Krull dimension of R. For a local ring $(R, \mathfrak{m})$, $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R.

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1. Gorenstein Modules

Here we collect some relevant basic facts concerning Gorenstein modules, which were introduced by R.Y. Sharp in [7] and studied extensively in [8], [9], [10], [11].

Definition 1.1. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d, and G a finitely generated R-module. We say that G is a* Gorenstein module (of type r) for *R provided*

$$\mu^i_R(\mathfrak{m}, G) = \begin{cases} 0 & \text{if } i \neq d \\ r & \text{if } i = d, \end{cases}$$

where $\mu^i_R(\mathfrak{m}, M) = \dim_k \operatorname{Ext}^i_R(k, M)$ is the i^{th} Bass number of M.

This definition is equivalent ([12, (1.2.5), (3.1.14), (1.2.15)]) to saying that G is a MCM R-module of finite injective dimension and type r. Also, existence of a Gorenstein R-module forces R to be CM ([10, (4.18)]). With this in mind, we see that a canonical module for R, if it exists, is precisely a Gorenstein module of type 1. If R does have a canonical module ω, then every Gorenstein module is isomorphic to a finite direct sum of copies of ω ([8, (4.6)]). The completion $\widehat{G}$ of a Gorenstein R-module G is a Gorenstein $\widehat{R}$-module, so we see that $\widehat{G} \cong \omega^r_{\widehat{R}}$, where r is the type of G and $\omega_{\widehat{R}}$ is the canonical module for $\widehat{R}$ (which exists, by Cohen's structure theorem).

We extend the definition of Gorenstein modules to non-local rings by saying that a finitely generated R-module G is Gorenstein if $G_\mathfrak{m}$ is a Gorenstein $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ of R.

As might be expected, Gorenstein modules enjoy a number of useful homological properties. Most of these can be seen by passing to the completion and using known properties of canonical modules, but we provide references here for completeness. The endomorphism ring $\operatorname{Hom}_R(G, G)$ is a free R-module of rank r^2, and $\operatorname{Ext}^i_R(G, G) = 0$ for $i > 0$ ([13, (3.1)]). For any MCM R-module M, $\operatorname{Hom}_R(M, G)$ is again MCM, and $\operatorname{Hom}_R(\operatorname{Hom}_R(M, G), G) \cong M^{r^2}$ ([11, (2.10)]). Finally, if $x \in \mathfrak{m}$ is an R-regular element, then x is also G-regular, and G/xG is a Gorenstein R/xR-module of the same type r, with $\operatorname{Hom}_R(G, G)/x \operatorname{Hom}_R(G, G) \cong \operatorname{Hom}_{R/xR}(G/xG, G/xG)$ ([10, (4.13)]).

2. Gorenstein Modules and Finite Index

Let $(R, \mathfrak{m})$ be a CM local ring. Following Ding [4], we define $\operatorname{index}(R)$ to be the least positive integer n so that $R/\mathfrak{m}^n$ is not a homomorphic image of a MCM module with no free summand. If no such n exists, say that $\operatorname{index}(R) = \infty$. Our goal in this section is to prove the following.

Theorem 2.1. *Let R be a CM local ring of dimension d. Assume that R has a Gorenstein module G. Then the following conditions are equivalent:*

1. $\operatorname{index}(R) < \infty$*;*
2. *R is Gorenstein on the punctured spectrum, that is, $R_{\mathfrak{p}}$ is Gorenstein for all primes $\mathfrak{p} \neq \mathfrak{m}$;*
3. *there exists an R-regular sequence $x_1, \dots, x_d$ such that $R/(x_1, \dots, x_d)$ is not a homomorphic image of a MCM R-module with no free summands;*
4. *every MCM R-module is a d^{th} syzygy.*

In particular, this recovers the following result of Ding for a CM local ring R with a canonical module: $\operatorname{index}(R) < \infty$ if and only if R is Gorenstein on the punctured spectrum.

Let R be a Noetherian ring, and let M be a finitely generated R-module. We have a natural map

$$\alpha_M : \operatorname{Hom}_R(M, R) \otimes_R M \longrightarrow \operatorname{Hom}_R(M, M)$$

given by $\alpha_M(f \otimes x)(y) = f(y)x$.

Lemma 2.2. *The image of α_M is equal to the set of all homomorphisms $h : M \longrightarrow M$ that factor through a free R-module.*

Proof. Let $h \in \operatorname{Im} \alpha_M$, so that there exist $f_i : M \longrightarrow R$ and $x_i \in M$, $i = 1, \dots, n$, such that $h(y) = \sum_1^n f_i(y) x_i$ for all $y \in M$. Then h factors through R^n, for if we let $g = [x_1, \cdots, x_n]$, and $f = [f_1 \cdots f_n]^t$, then $h = gf$.

For the other inclusion, suppose we have $h : M \longrightarrow M$ factoring through R^n as $h = gf$. Let $g_1, \dots, g_n$ be the restrictions of g to each copy of R. Let f_j be the composition of f with the projection to the j^{th} component of R^n. Then we have $h = \sum_{j=1}^n g_j f_j : M \longrightarrow M$. Let $x_i = g_i(1)$. Then for any $y \in M$,

$$h(y) = \sum_{j=1}^n f_i(y) x_i = \alpha_M \big(\sum_{j=1}^n f_i \otimes x_i \big)(y).$$

□

For the next four results, assume that $(R,\mathfrak{m})$ is a CM local ring and G is a Gorenstein R-module of minimum type r. We may consider R as a submodule of $\operatorname{Hom}_R(G,G)$ via the map sending x to "multiplication by x". As G is a faithful R-module ([7, 4.2]), this map is injective. Define another submodule τ of $\operatorname{Hom}_R(G,G)$ by $\tau = R \cap \operatorname{Im}\alpha_G$. Then τ can be considered either as a submodule of $\operatorname{Hom}_R(G,G)$ or as an ideal of R. We point out that in the case $r=1$, τ coincides with the trace of G in R. In general the two are distinct. However, τ does share the following important property with the trace:

Lemma 2.3. *The ring R is Gorenstein if and only if $\tau = R$.*

Proof. If R is Gorenstein, then since G has minimum type, we have $G \cong R$, and every endomorphism of G is given by multiplication by some element of R. Conversely, if $\tau = R$, then in particular the identity endomorphism of G is in τ. That is, G is isomorphic to a direct summand of a free module (Lemma 3.2), hence free. Therefore R is a direct summand of G and so is a Gorenstein ring. □

This gives the following fundamental observation:

Proposition 2.4. *Let $\mathfrak{p} \in \operatorname{Spec} R$. Then $R_\mathfrak{p}$ is Gorenstein if and only if $\tau R_\mathfrak{p} = R_\mathfrak{p}$. In particular, R is Gorenstein on the punctured spectrum if and only if τ is an $\mathfrak{m}$-primary ideal.*

We now consider the functor on R-modules given by $M^\vee = \operatorname{Hom}_R(M,G)$. If R is known to have a canonical module ω, we will write M' for the usual canonical dual $\operatorname{Hom}_R(M,\omega)$ of M.

Note 2.5: We have $\widehat{M^\vee} \cong ((\widehat{M})')^r$ for every finitely generated R-module M. Indeed, passing to the completion $\widehat{R}$ and letting $\omega = \omega_{\widehat{R}}$, we have

$$\widehat{M^\vee} = \operatorname{Hom}_R(M,G) \otimes_R \widehat{R} = \operatorname{Hom}_{\widehat{R}}(\widehat{M},\omega^r) = \operatorname{Hom}_{\widehat{R}}(\widehat{M},\omega)^r.$$

In what follows, we will often wish to say that one R-module, M, is isomorphic to a direct summand of another, N. We will denote this situation by $M \mid N$.

Let $x \in \mathfrak{m}$ be an R-regular, and so G-regular, element, and set $\overline{R} = R/xR$, $\overline{G} = G/xG$. Let $\operatorname{syz}_R(M)$ denote the first syzygy of M in a minimal R-free resolution. Note that $Z := \operatorname{syz}_R(\overline{G})$ is a MCM R-module; in particular, we have $\widehat{Z}'' \cong \widehat{Z}$.

The next lemma, with $G = \omega$, is [4, Lemma 1.6]. Our proof is by reduction to that case.

Lemma 2.6. *With notation as above,* $\mathrm{syz}_R(\overline{G})^\vee$ *has a nonzero free direct summand if and only if* $x \in \tau$*.*

Proof. Put $Z = \mathrm{syz}_R(\overline{G})$, and suppose that $R \mid Z^\vee$. Then $\widehat{R} \mid (\widehat{Z}')^r$. By the Krull-Schmidt uniqueness theorem for $\widehat{R}$-modules, $\widehat{R} \mid \widehat{Z}'$. Let ω be the canonical module for $\widehat{R}$ (which exists, *cf.* [12, (3.3.8)]). Dualizing into ω, we see that $\omega \mid \widehat{Z}$. But $\widehat{Z} \cong \mathrm{syz}_{\widehat{R}}(\omega^r/x\omega^r) \cong \mathrm{syz}_{\widehat{R}}(\overline{\omega})^r$, where $\overline{\omega} := \omega/x\omega$. Another application of the Krull-Schmidt theorem shows that $\omega \mid \mathrm{syz}_{\widehat{R}}(\overline{\omega})$. By [4, (1.6)], then, the endomorphism of ω given by multiplication by x factors through a free $\widehat{R}$-module. Since $\widehat{G} \cong \omega^r$, the corresponding map on $\widehat{G}$ also factors through a free $\widehat{R}$-module. Hence, by Lemma 2.3, $x \in \tau\widehat{R}$, and so $x \in \tau$.

For the other implication, suppose $x \in \tau$. Then $x \in \tau\overline{R}$, so $\mathrm{syz}_R(\overline{\omega})$ has a direct summand isomorphic to ω by [4, Lemma 1.6]. Then $\mathrm{syz}_R(\overline{G})$ has a direct summand isomorphic to G, and $\mathrm{syz}_R(\overline{G})^\vee$ has a free summand. □

Proof of Theorem 2.1. If $d = 0$, there is nothing to prove, so assume $d > 0$. We begin with a construction which will be used in the proof.

Let $x \in \mathfrak{m}$ be an arbitrary R-regular element, and let

$$0 \longrightarrow \mathrm{syz}_R(\overline{G}) \longrightarrow R^m \longrightarrow \overline{G} \longrightarrow 0$$

be the first part of a minimal free resolution of $\overline{G} := G/xG$. Then applying $\mathrm{Hom}_R(-, G)$ gives an exact sequence

$$0 \longrightarrow G^m \longrightarrow \mathrm{syz}_R(\overline{G})^\vee \longrightarrow \mathrm{Ext}^1_R(\overline{G}, G) \longrightarrow 0.$$

It is elementary from the properties of Gorenstein modules (Section 1) that $\mathrm{Ext}^1_R(\overline{G}, G) \cong \overline{R}^{r^2}$, where $\overline{R} := R/(x)$. Now, since $\mathrm{syz}_R(\overline{G})$ is a MCM R-module, its G-dual $\mathrm{syz}_R(\overline{G})^\vee$ is also MCM. This gives the short exact sequence

$$0 \longrightarrow G^m \longrightarrow \mathrm{syz}_R(\overline{G})^\vee \longrightarrow \overline{R}^{r^2} \longrightarrow 0, \qquad (*)$$

with the middle term a MCM R-module.

(1) $\implies$ (2). We assume that R is not Gorenstein on the punctured spectrum. Fix an arbitrary positive integer n. Since the ideal τ is not $\mathfrak{m}$-primary (Proposition 2.4), $\mathfrak{m}^n \not\subseteq \tau$. Choose a regular element $x \in \mathfrak{m}^n - \tau$. Then Lemma 2.6 implies that $\mathrm{syz}_R(\overline{G})^\vee$ has no nonzero free summands. So in $(*)$, we have constructed a MCM R-module with no nonzero free summands which maps onto R/xR. In turn, since $x \in \mathfrak{m}^n$, R/xR maps onto $R/\mathfrak{m}^n$. As n was arbitrary, this shows that $\mathrm{index}(R)$ is infinite.

(2) $\implies$ (3). By Proposition 2.4, the ideal τ is $\mathfrak{m}$-primary. Thus there exists an R-sequence $x_1, \dots , x_d$ in τ, where, as before, $d = \dim(R)$. We use induction on d to show that this is the desired R-sequence.

For the case $d = 1$, set $x = x_1$ and $\overline{R} = R/(x)$. As before, we obtain the short exact sequence $(*)$. By Lemma 2.6, since $x \in \tau$, $\operatorname{syz}_R(\overline{G})^\vee \cong U \oplus R$ for some R-module U. Denote the map $U \oplus R \longrightarrow \overline{R}^{r^2}$ in $(*)$ by f. We claim first that $f(U) \neq \overline{R}^{r^2}$. If $f(U) = \overline{R}^{r^2}$, we have the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U & \longrightarrow & f(U) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^m & \longrightarrow & U \oplus R & \xrightarrow{f} & \overline{R}^{r^2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{\cong} & R & \longrightarrow & 0 & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array} \tag{$**$}$$

By the Snake Lemma, then, $G^m \longrightarrow R$ is surjective, so a split surjection, and so R is Gorenstein. In this case, $G \cong R$ and $U = 0$, a contradiction. This shows that $f(U) \neq \overline{R}^{r^2}$. Now suppose that $g : Z \longrightarrow \overline{R}$ is a surjection with Z a MCM R-module. This gives a surjection $Z^{r^2} \longrightarrow \overline{R}^{r^2}$, which we also call g. Since $\operatorname{Ext}^1_R(Z^{r^2}, G^m) = 0$, g lifts to $h : Z^{r^2} \longrightarrow U \oplus R$ such that $g = fh$. We claim that the composition $\pi h : Z^{r^2} \longrightarrow U \oplus R \longrightarrow R$ is surjective. Suppose $\pi h(Z^{r^2}) \subseteq \mathfrak{m}$, and let $a \in \overline{R}^{r^2}$. Write $a = g(z) = f(h(z))$ for some $z \in Z^{r^2}$. Also write $h(z) = (u, r) \in U \oplus R$. Then $r \in \mathfrak{m}$ by assumption, so $a = rf(0, 1) + f(u, 0) \in \mathfrak{m}\overline{R}^{r^2} + f(U)$. Since a was arbitrary, this shows that $\overline{R}^{r^2} = \mathfrak{m}\overline{R}^{r^2} + f(U)$, and Nakayama's lemma implies $f(U) = \overline{R}^{r^2}$, a contradiction. The surjection $\pi h : Z^{r^2} \longrightarrow R$ shows that Z has a free summand, as desired.

Now suppose $d > 1$ and there exists a surjection $Z \longrightarrow R/(x_1, \dots, x_d)$ with Z MCM. Then $\overline{Z} \longrightarrow \overline{R}/(\overline{x_2}, \dots, \overline{x_d})$ is also a surjection, where a bar indicates reduction modulo x_1. Since $\overline{x_2}, \dots, \overline{x_d}$ are in $\overline{\tau}$, $\overline{Z}$ has an $\overline{R}$-summand by the case $d = 1$. But then $Z \longrightarrow \overline{R}$ is surjective and applying the case $d = 1$ again shows that Z has a free summand.

(3) $\implies$ (1). The ideal $(x_1, \dots, x_d)$ is $\mathfrak{m}$-primary, so $\mathfrak{m}^n \subseteq (x_1, \dots, x_d)$ for some $n > 1$. Then the surjection $R/\mathfrak{m}^n \longrightarrow R/(x_1, \dots, x_d)$ shows that no MCM R-module without free summands maps onto $R/\mathfrak{m}^n$, that is, $\operatorname{index}(R) < \infty$.

(2) $\implies$ (4). This is what is proved in [5, (3.8)], but since the statement there is incorrect we review the argument here. We may assume that $d \geq 3$ by Theorems 3.5 and 3.6 of [5]. Then M is reflexive by [5, (3.6)]. Resolve $M^* = \operatorname{Hom}_R(M, R)$:

$$\cdots \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M^* \longrightarrow 0$$

and dualize, obtaining a complex

$$0 \longrightarrow M \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \cdots \longrightarrow F_{d-1}^* \longrightarrow C \longrightarrow 0, \qquad (\dagger)$$

where $C = \operatorname{coker}(F_{d-2}^* \longrightarrow F_{d-1}^*)$. It will suffice to show that (†) is exact.

If (†) is not exact choose i minimal, $1 \leq i \leq d-2$, such that (†) is not exact at F_i^*. Let D be the cokernel of $F_{i-1}^* \longrightarrow F_i^*$. Then D contains a submodule isomorphic to $\operatorname{Ext}_R^i(M^*, R)$. For any prime $\mathfrak{p} \neq \mathfrak{m}$, $\operatorname{Ext}_{R_\mathfrak{p}}^i(M_\mathfrak{p}^*, R_\mathfrak{p}) = 0$, since $R_\mathfrak{p}$ is Gorenstein. Thus $\operatorname{Ext}_R^i(M^*, R)$ is a nonzero module of finite length, whence $\operatorname{depth} D = 0$. The Depth Lemma then implies that $\operatorname{depth} M \leq i + 1 < d$, a contradiction.

(4) $\implies$ (2). Since every MCM R-module is a d^{th} syzygy, there exists an exact sequence

$$0 \longrightarrow G \longrightarrow F_d \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow X \longrightarrow 0$$

with each F_i a free R-module. Let M be the cokernel of $G \longrightarrow F_d$, the $(d-1)^{\text{th}}$ syzygy. Let $\mathfrak{p} \neq \mathfrak{m}$ be given, and localize at $\mathfrak{p}$. Then $M_\mathfrak{p}$ is a $(d-1)^{\text{th}}$ syzygy over the CM ring $R_\mathfrak{p}$, which has dimension $\leq d-1$, so $M_\mathfrak{p}$ is MCM. Since $\operatorname{Ext}^1_{R_\mathfrak{p}}(M_\mathfrak{p}, G_\mathfrak{p}) = 0$, the sequence $0 \longrightarrow G_\mathfrak{p} \longrightarrow (F_d)_\mathfrak{p} \longrightarrow M_\mathfrak{p} \longrightarrow 0$ splits. Hence $G_\mathfrak{p}$ is free, and $R_\mathfrak{p}$ is Gorenstein.

□

Remark 2.7: The implication (2) $\implies$ (4) does not require the existence of a canonical or Gorenstein module. It would be interesting to find a proof of the other implications, especially (4) $\implies$ (2) that also did not use Gorenstein modules.

Theorem 2.1 generalizes easily to the case of a non-CM local ring. We leave the details of the proof to the reader.

Proposition 2.8. *Let R be a local ring satisfying Serre's condition (S_k). Assume that G is an R-module such that $G_\mathfrak{p}$ is a Gorenstein $R_\mathfrak{p}$-module for all primes of height less than or equal to k. Then the following conditions are equivalent:*

1. $\operatorname{index}(R) < \infty$ *for all primes of height* $\leq k$*;*
2. *R is Gorenstein in codimension $k-1$, that is, $R_\mathfrak{p}$ is Gorenstein for all primes $\mathfrak{p}$ of height $\leq k-1$;*
3. *when $\mathfrak{p}$ has height $\leq k$, every MCM $R_\mathfrak{p}$-module is a k^{th} syzygy;*
4. *every R-module satisfying (S_k) is a k^{th} syzygy.*

3. Application to Rings of Finite Cohen–Macaulay Type

In this section, we apply a result of R. Guralnick, reproduced for convenience, to show that a CM local ring of finite CM type has finite index. This lets us prove that a CM local ring of finite CM type with a Gorenstein module is Gorenstein on the punctured spectrum. As mentioned in the Introduction, R. Wiegand and I have shown that when R is excellent, finite CM type actually implies that R is regular on the punctured spectrum.

Lemma 3.1 ([14, Cor.2]). *Let $(R, \mathfrak{m})$ be a local ring and let M and N be finitely generated R-modules. If $N/\mathfrak{m}^n N$ is isomorphic to a direct summand of $M/\mathfrak{m}^n M$ for every $n \gg 0$, then N is isomorphic to a direct summand of M.*

Definition 3.2. *A CM local ring $(R, \mathfrak{m})$ has* finite Cohen–Macaulay type *(or finite*

CM type) if R has, up to isomorphism, only finitely many indecomposable MCM modules.

Theorem 3.3. *Let $(R, \mathfrak{m})$ be a CM local ring of finite CM type. Assume that R has a Gorenstein module G. Then R has finite index.*

Proof. Let $\{M_1, \dots, M_r\}$ be a complete set of representatives for the isomorphism classes of nonfree indecomposable MCM R-modules. Since no M_i has a nonzero free summand, there exist integers n_i, $1 \le i \le r$, so that for $s \ge n_i$, $R/\mathfrak{m}^s$ is not a direct summand of$M_i/\mathfrak{m}^s M_i$. Then for $s \ge n_i$, there exists no surjection $M_i \longrightarrow R/\mathfrak{m}^s$ (Lemma 3.1). Set $N = \max\{n_i\}$. Let X be any MCM R-module without nonzero free summands, and decompose $X = M_1^{a_1} \oplus \cdots \oplus M_r^{a_r}$. If there were a surjection $X \longrightarrow R/\mathfrak{m}^N$, then, since R is local, one of the summands M_i would map onto $R/\mathfrak{m}^N$, contradicting the choice of N. As X was arbitrary, this shows that $\operatorname{index}(R) < \infty$. □

This, together with Theorem 2.1, immediately gives the following corollary.

Corollary 3.4. *Let $(R, \mathfrak{m})$ be a CM local ring of finite CM type. Assume that R has a Gorenstein module. Then R is Gorenstein on the punctured spectrum.*

4. Open Gorenstein Locus

For a given finitely generated R-module M, we may consider the prime ideals $\mathfrak{p}$ of R such that $M_\mathfrak{p}$ is a Gorenstein $R_\mathfrak{p}$-module. Our task in this section is to prove that for a large class of rings containing all excellent rings (in fact all acceptable rings, cf. [17]), these primes form a Zariski-open subset of $\operatorname{Spec} R$. The appropriate hypothesis is a Nagata-type criterion: that the Gorenstein locus of $R/\mathfrak{p}$ contain a nonempty open subset of $\operatorname{Spec}(R/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

First we collect some preliminary lemmas. The first appears as Lemma 1.1 of [15].

Lemma 4.1. *Let R be a Noetherian ring and M a finitely generated R-module. Let $\mathfrak{p}$ be a minimal element of $\operatorname{Ass}_R(M)$. Then there exist an element $f \notin \mathfrak{p}$ and a filtration $(U_i)_{0 \le i \le m}$ for the R_f-module M_f such that for all i, $0 \le i \le m-1$, $U_i/U_{i+1} \cong R_f/\mathfrak{p}_f$.*

Lemma 4.2. *Let R be a Noetherian ring, and M a finitely generated R-module. Suppose that $\mathfrak{p}$ is the unique associated prime of R, and that there exists an integer n such that $\operatorname{Ext}_R^n(R/\mathfrak{p}, M) = 0$. Then there exists an element $f \notin p$ so that*

$\operatorname{Ext}^i_{R_f}(R_f/\mathfrak{p}_f, M_f) = 0$ *for all* $i \geq n$.

Proof. If $\mathfrak{p} = (0)$, then $\operatorname{Ext}^i_R(R/\mathfrak{p}, M) = 0$ for all $i > 0$, so there is nothing to prove. Hence we may assume $\mathfrak{p} \neq (0)$. Every associated prime of $\mathfrak{p}$ is an associated prime of R, so by the assumption on $\mathfrak{p}$, $\operatorname{Ass}_R(\mathfrak{p}) = \{\mathfrak{p}\}$. By Lemma 4.1, there exist $f \notin \mathfrak{p}$ and a filtration $(U_i)_{0 \leq i \leq m}$ for the R_f-module $\mathfrak{p}_f$ such that $U_0 = \mathfrak{p}_f$, $U_m = 0$, and for each $i \leq m-1$, $U_i/U_{i+1} \cong R_f/\mathfrak{p}_f$. Assume, inductively, that $\operatorname{Ext}^i_{R_f}(R_f/\mathfrak{p}_f, M_f) = 0$ for some $i \geq n$. Since $\mathfrak{p}_f$ is filtered by the modules $R_f/\mathfrak{p}_f$ it follows that $\operatorname{Ext}^i_{R_f}(\mathfrak{p}_f, M_f) = 0$. But then $\operatorname{Ext}^{i+1}_{R_f}(R_f/\mathfrak{p}_f, M_f) = 0$, and the induction is complete. □

Lemma 4.3. *Let* $(R, \mathfrak{m}, k)$ *be a local ring and* G *a finitely generated* R*-module. Suppose that* $\mathfrak{p} \in \operatorname{Spec} R$ *is such that* $\dim R/\mathfrak{p} = \dim R = d$, $\operatorname{Ext}^i_R(R/\mathfrak{p}, G) = 0$ *for all* $i > 0$, *and* $\operatorname{Hom}_R(R/\mathfrak{p}, G) \cong (R/\mathfrak{p})^r$. *Then* G *is a Gorenstein module of type* r *for* R *if and only if* $R/\mathfrak{p}$ *is a Gorenstein ring.*

Proof. Let $I^\bullet$ be a minimal R-injective resolution of G. Then $\operatorname{Ext}^\bullet_R(R/\mathfrak{p}, G)$ is the homology of the complex $\operatorname{Hom}_R(R/\mathfrak{p}, I^\bullet)$. Therefore $\operatorname{Hom}_R(R/\mathfrak{p}, I^\bullet)$ is an $R/\mathfrak{p}$-injective resolution of $(R/\mathfrak{p})^r$. Now $\operatorname{Hom}_R(-, I^\bullet)$ and $\operatorname{Hom}_{R/\mathfrak{p}}(-, \operatorname{Hom}_R(R/\mathfrak{p}, I^\bullet))$ are naturally equivalent functors of $R/\mathfrak{p}$-modules. Hence

$$\begin{aligned}
\operatorname{Ext}^i_R(k, G) &= H^i(\operatorname{Hom}_R(k, I^\bullet)) \\
&\cong H^i(\operatorname{Hom}_{R/\mathfrak{p}}(k, \operatorname{Hom}_R(R/\mathfrak{p}, I^\bullet))) \\
&= \operatorname{Ext}^i_{R/\mathfrak{p}}(k, (R/\mathfrak{p})^r) \\
&\cong \operatorname{Ext}^i_{R/\mathfrak{p}}(k, R/\mathfrak{p})^r.
\end{aligned}$$

The conclusion follows immediately. □

Definition 4.4. *Let* R *be a Noetherian ring and* M *a finitely generated* R*-module. The Gorenstein locus of* M, *denoted* $\Omega_R(M)$, *is the subset* $\{\mathfrak{p} \in \operatorname{Spec}(R) : M_\mathfrak{p}$ *is a Gorenstein* $R_\mathfrak{p}$*-module*$\}$ *of* $\operatorname{Spec}(R)$. *The Gorenstein locus of* R, *denoted* $\operatorname{Gor}(R)$, *is the set* $\{\mathfrak{p} \in \operatorname{Spec}(R) : R_\mathfrak{p}$ *is Gorenstein*$\}$. *Finally, the regular locus of* R, $\operatorname{Reg}(R)$, *is the set* $\{\mathfrak{p} \in \operatorname{Spec}(R) : R_\mathfrak{p}$ *is regular*$\}$.

Theorem 4.5. *Let* R *be a Noetherian ring and* M *a finitely generated* R*-module. Assume that for every* $\mathfrak{p} \in \Omega_R(M)$, $\operatorname{Gor}(R/\mathfrak{p})$ *contains a nonempty open subset of* $\operatorname{Spec}(R/\mathfrak{p})$. *Then* $\Omega_R(M)$ *is open in* $\operatorname{Spec}(R)$.

Proof. Since $\Omega_R(M)$ is stable under generization, we need only show that $\Omega_R(M)$ is open in the constructible topology on $\operatorname{Spec}(R)$ ([16, p.46]). In other words, it is enough to show that for each $\mathfrak{p} \in \operatorname{Spec}(R)$, $\Omega_R(M) \cap V(\mathfrak{p})$ is an open subset of $V(\mathfrak{p})$.

Step 1. We may assume that $\mathfrak{p}$ is the unique associated prime of R. Indeed, since $R_\mathfrak{p}$ has a Gorenstein module, $R_\mathfrak{p}$ is Cohen–Macaulay by [10], so $\mathfrak{p}$ contains an $R_\mathfrak{p}$-regular sequence $\underline{x} = (x_1, \ldots, x_h)$, where $h = \operatorname{height} \mathfrak{p}$. Choose $a \notin \mathfrak{p}$ so that $\underline{x}_a$ is R_a-regular. Replace R by $R_a/\underline{x}_a$ and $\mathfrak{p}$ by $\mathfrak{p}_a/\underline{x}_a$, to assume that $\mathfrak{p}$ is a minimal prime of R. Then let b be an element of all the associated primes of R except $\mathfrak{p}$. Inverting b yields a ring R_b in which $\mathfrak{p}_b$ is the unique associated prime. Reset notation, replacing R by R_b and $\mathfrak{p}$ by $\mathfrak{p}_b$, so that now $\operatorname{Spec}(R) = V(\mathfrak{p})$. Note that inverting more elements outside $\mathfrak{p}$ will preserve the fact that $\mathfrak{p}$ is the unique associated prime of R.

Step 2. We may assume that $\operatorname{Ext}^i_R(R/\mathfrak{p}, M) = 0$ for all $i > 0$. As $R_\mathfrak{p}$ is zero-dimensional and $M_\mathfrak{p}$ is a Gorenstein module for $R_\mathfrak{p}$, $\operatorname{Ext}^1_R(R/\mathfrak{p}, M)_\mathfrak{p} = 0$. Hence we can invert a single element $c \notin \mathfrak{p}$ so that $\operatorname{Ext}^1_{R_c}(R_c/\mathfrak{p}_c, M_c) = 0$. Since $\mathfrak{p}_c$ is the unique associated prime of R_c, by Lemma 4.2 we may invert another element $d \notin \mathfrak{p}$ to assume that all the higher Ext's vanish as well. Reset notation.

Step 3. We may assume that $\operatorname{Hom}_R(R/\mathfrak{p}, M) \cong (R/\mathfrak{p})^r$, for some integer $r \geq 1$, as follows. By assumption, $M_\mathfrak{p}$ is a Gorenstein module for $R_\mathfrak{p}$ of type, say, r. Then $\operatorname{Hom}_R(R/\mathfrak{p}, M)_\mathfrak{p} \cong (\kappa(\mathfrak{p}))^r$, where $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ (recall that $R_\mathfrak{p}$ is 0-dimensional). Choose a homomorphism of R-modules $\varphi : \operatorname{Hom}_R(R/\mathfrak{p}, M) \longrightarrow (R/\mathfrak{p})^r$ inducing an $R_\mathfrak{p}$-isomorphism $\operatorname{Hom}_R(R/\mathfrak{p}, M)_\mathfrak{p} \longrightarrow (\kappa(\mathfrak{p}))^r$. As usual, there exists $e \notin \mathfrak{p}$ so that φ_e is an isomorphism. Replace R by R_e and continue.

Step 4. We may assume that $R/\mathfrak{p}$ is Gorenstein. In fact, the Gorenstein locus of $R/\mathfrak{p}$ contains a nonempty open set by hypothesis, so we may invert a single element $f \notin \mathfrak{p}$ so that $R_f/\mathfrak{p}_f$ is Gorenstein.

Step 5. We may assume that R is local. Both the Gorenstein property of the ring R and the condition of being a Gorenstein module are locally defined.

We have reduced our problem to showing that if $(R, \mathfrak{m})$ is a local ring and $\mathfrak{p} \in \operatorname{Spec}(R)$ is the unique associated prime of R, with $\operatorname{Hom}_R(R/\mathfrak{p}, M) \cong (R/\mathfrak{p})^r$, $\operatorname{Ext}^i_R(R/\mathfrak{p}, M) = 0$ for $i > 0$, and $R/\mathfrak{p}$ Gorenstein, then M is a Gorenstein R-module of type r. This is the content of Lemma 4.3, which finishes the proof. □

Corollary 4.6. *Let R be an acceptable ring and M a finitely generated R-module. Then $\Omega_R(G)$ is open in* $\operatorname{Spec}(R)$.

Proof. By the definition of an acceptable ring ([17]), $\operatorname{Gor}(S)$ is open in $\operatorname{Spec}(S)$ for every finitely generated R-algebra S. Apply Theorem 4.5. □

Corollary 4.7. *Let R be an excellent ring and M a finitely generated R-module. Then $\Omega_R(M)$ is open in* $\operatorname{Spec}(R)$.

Proof. By [15, 1.5], an excellent ring is acceptable. Corollary 4.6 gives the result. □

Acknowledgment

This work formed part of my doctoral dissertation at the University of Nebraska. It was made possible in part by the support of the Maude Hammond Fling Graduate Fellowship. I am grateful to my advisor, Roger Wiegand, for belaying. I also thank Craig Huneke, Luchezar Avramov, and Mark Walker for their patience and encouragement in conversations.

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