NON-COMMUTATIVE DESINGULARIZATION OF DETERMINANTAL VARIETIES I

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Abstract. We show that determinantal varieties defined by maximal minors of a generic matrix have a non-commutative desingularization, in that we construct a maximal Cohen-Macaulay module over such a variety whose endomorphism ring is Cohen-Macaulay and has finite global dimension. In the case of the determinant of a square matrix, this gives a non-commutative crepant resolution.

Contents

1. Introduction

Let K be a field and $X = (x_{ij})$ an $(m \times n)$-matrix of indeterminates over K having $n \geqslant m$. With $S = K[x_{ij}]$ the polynomial ring in the x_{ij}, the matrix X determines the *generic S-linear map* $\varphi\colon S^n \longrightarrow S^m$. Let $\operatorname{Spec} R$ be the locus in $\operatorname{Spec} S$ where φ has non-maximal rank; equivalently R is the quotient of S given by the maximal minors of X.

The classical R-modules $M_a = \operatorname{cok} \bigwedge_S^a \varphi$ are familiar objects in commutative algebra. In particular it is known that they are maximal Cohen–Macaulay and are resolved by the Buchsbaum-Rim complex ([7, Corollary 2.6], see also [23]). In this paper we show that the $(M_a)_a$ conspire to yield a kind of *non-commutative desingularization* of the singular variety $\operatorname{Spec} R$. More precisely we prove the following result.

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Theorem A (Thm. 6.5). *For $1 \leqslant a \leqslant m$ put $M_a = \operatorname{cok} \bigwedge_S^a \varphi$ and $M = \bigoplus_a M_a$. Then the endomorphism algebra $E = \operatorname{End}_R(M)$ is maximal Cohen–Macaulay as an R-module, and has moreover finite global dimension.*

If $m = n$ then R is the hypersurface ring $R = S/(\det \varphi)$ and hence R is Gorenstein. In this case our non-commutative desingularization is an example of a *non-commutative crepant resolution* as defined in [21]. Non-commutative desingularizations occurred probably first in theoretical physics (e.g. [2]) but they have recently been encountered in a number of purely mathematical contexts (e.g. [3, 14, 15, 17, 20]).

The next result is a description by generators and relations of the non-commutative resolution E.

Theorem B (Rem. 7.6, Thm. 7.17). *As a K-algebra, E is isomorphic to the path algebra $K\widetilde{\mathsf{Q}}$ of the quiver*

$$\widetilde{\mathsf{Q}}: \quad \begin{tikzcd}[column sep=huge] 1 \arrow[r, shift right, swap, "g_1 \,\vdots\, g_n"] & 2 \arrow[l, shift right, swap, "\lambda_1 \,\vdots\, \lambda_m"] \arrow[r, shift right, swap, "g_1 \,\vdots\, g_n"] & \cdots\cdots \arrow[l, shift right, swap, "\lambda_1 \,\vdots\, \lambda_m"] \arrow[r, shift right, swap, "g_1 \,\vdots\, g_n"] & m \arrow[l, shift right, swap, "\lambda_1 \,\vdots\, \lambda_m"] \end{tikzcd}$$

modulo relations

$$\begin{aligned} \lambda_i\lambda_j + \lambda_j\lambda_i &= 0 = \lambda_i^2 && \text{for } i,j = 1,\dots,m;\\ g_ig_j + g_jg_i &= 0 = g_i^2 && \text{for } i,j = 1,\dots,n;\\ \lambda_k(\lambda_ig_j + g_j\lambda_i) &= (\lambda_ig_j + g_j\lambda_i)\lambda_k && \text{for } i,k = 1,\dots,m,\ j = 1,\dots,n; \text{ and}\\ g_l(\lambda_ig_j + g_j\lambda_i) &= (\lambda_ig_j + g_j\lambda_i)g_l && \text{for } i = 1,\dots,m,\ j,l = 1,\dots,n. \end{aligned}$$

(terms in those relations which go outside the quiver are silently suppressed, see §7.5).

Despite the fact that Theorems A and B have purely algebraic statements, we will prove them by relying on algebraic geometry. In our proofs we use the classical fact that $\operatorname{Spec} R$ has a *Springer type resolution of singularities*. To be precise, define the incidence variety

$$\mathcal{Z} = \{\, ([\lambda], \theta) \in \mathbb{P}^{m-1}(K) \times M_{m\times n}(K) \mid \lambda\theta = 0 \,\}$$

with projections $p' \colon \mathcal{Z} \longrightarrow \mathbb{P}^{m-1}$ and $q' \colon \mathcal{Z} \longrightarrow \operatorname{Spec} R$. The following theorem contains the key geometric facts we use.

Theorem C (Thm. 6.2, Thm. 6.4, Thm. 6.5). *The scheme $\mathcal{Z}$ is projective over* $\operatorname{Spec} R$*, which is of finite type over K. The $\mathcal{O}_{\mathcal{Z}}$-module*

$$\mathcal{T} := p'^{*}\left(\bigoplus_{a=1}^{m} \left(\bigwedge\nolimits^{a-1} \Omega_{\mathbb{P}^{m-1}}\right)(a)\right)$$

is a classical tilting bundle on $\mathcal{Z}$ in the sense of [13]*, i.e.*

(1) $\mathcal{T}$ is a locally free sheaf, in particular, a perfect complex on $\mathcal{Z}$,
(2) $\mathcal{T}$ generates the derived category $\mathcal{D}(\operatorname{Qch}(\mathcal{Z}))$, in that $\operatorname{Ext}^{\bullet}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T}, C) = 0$ for a complex C in $\mathcal{D}(\operatorname{Qch}(\mathcal{Z}))$ implies $C \cong 0$, and
(3) $\operatorname{Hom}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T}, \mathcal{T}[i]) = 0$ for $i \neq 0$.

Furthermore we have

(3) $M \cong \mathbf{R}q'_*\mathcal{T}$, *and*
(4) $E \cong \operatorname{End}_{\mathcal{Z}}(\mathcal{T})$.

This theorem implies in particular that the geometric resolution $\mathcal{Z}$ and the non-commutative resolution E are derived equivalent [19]. Hence $\mathcal{Z}$ parametrizes certain objects in the derived category of E. The following result gives a more precise interpretation of this idea.

Theorem D (Thm. 8.9). *The variety $\mathcal{Z}$ is the fine moduli space for the $\widetilde{\mathsf{Q}}$-representations W of dimension vector $(1, m-1, \binom{m-1}{2}, \ldots, 1)$ that are generated by the last component W_m.*

The proof of Theorem C is based on the explicit (and characteristic-free) computation of the cohomology of certain homogeneous bundles on $\mathbb{P}^{m-1}$. More precisely, for

$$\mathcal{M}_a^b = \mathscr{H}om_{\mathcal{O}_{\mathbb{P}^{m-1}}}\left((\textstyle\bigwedge^{b-1}\Omega)(b), (\textstyle\bigwedge^{a-1}\Omega)(a)\right)$$

we compute in Theorem 3.9 the cohomology of $\mathcal{M}_a^b(-c)$ for $c \in \mathbb{Z}$. (The interested reader may wish to compare the Appendix by Weyman to [10], which, by different methods, computes as a special case $\operatorname{Ext}^i(\bigwedge^p\Omega, \bigwedge^q\Omega)$ for all $i \geqslant 0$.) This result is used in Theorem 5.3 to compute the shape of the minimal S-projective resolution of $q'_*p'^*\mathcal{M}_a^b(-c)$ in many cases. This yields in particular a large supply of maximal Cohen-Macaulay R-modules.

To prove Theorem 5.3 we use a new "degeneracy criterion for sparse spectral sequences" (see Proposition 4.4) which we think is interesting in its own right. Under mild boundedness hypotheses this result asserts that if a page of a spectral sequence has projective entries then we can obtain from it a projective resolution of its limit.

Two additional results occupy the last two sections: In §9 we give an explicit minimal S-free presentation for the maximal Cohen-Macaulay R-modules $\operatorname{Hom}_R(M_a, M_b)$ in terms of certain minors in X, and in §10 we compute (in characteristic zero) the shape of the minimal graded free resolution of the graded simples of E.

In characteristic zero we know how to generalize most of our results to arbitrary determinantal varieties. This will be covered in a sequel to the current paper. The present paper is largely characteristic-free.

2. Notation

Symbol	**Meaning**
K	a commutative base ring, most often a field
F, G	projective K-modules of finite ranks $m \leqslant n$
$\bigwedge^a F$, F_a	indicated exterior power of F
$\lvert F\rvert$	determinant of F, $\bigwedge^{\operatorname{rank} F} F$
$\mathbb{S}^b = \operatorname{Sym}_K^b$	symmetric power
H	$\operatorname{Hom}_K(G, F)$
S	$\mathbb{S}(H^\vee)$, a polynomial algebra over K
$-^\vee$	dual over K or S, as context implies
A, $\mathsf{K}^-(\mathbb{P}\mathsf{A})$	abelian category with enough projectives and the homotopy category of its projectives

$\mathcal{G}, \mathcal{F}$	free S-modules induced from G and F
$\varphi\colon \mathcal{G} \longrightarrow \mathcal{F}$	the generic S-linear map
$X = (x_{ij})$	generic $(m \times n)$-matrix of local coordinates on $\operatorname{Spec} S$
R	the quotient of S determined by the maximal minors of X
$\mathbb{P} = \mathbb{P}(F^\vee)$	K-projective space on the dual of F
$\pi\colon \mathbb{P} \longrightarrow K$	structure morphism
$\mathcal{Y}$	$\mathbb{P} \times_{\operatorname{Spec} K} H$, with projections $p\colon \mathcal{Y} \longrightarrow \mathbb{P}$, $q\colon \mathcal{Y} \longrightarrow H$
$\mathcal{Z}$	the incidence variety desingularizing $\operatorname{Spec} R$, with inclusion $j\colon \mathcal{Z} \longrightarrow \mathcal{Y}$
coh and Qch	categories of coherent and quasi-coherent sheaves
$\mathcal{D}^b_f$	bounded derived category of complexes with finite homology
$\mathbb{K}(\mathrm{id}_F)$	affine tautological Koszul complex over id_F
$\mathbb{K}$	projective tautological Koszul complex
F^b_a	$\operatorname{Hom}_K(F_b, F_a) = F_a \otimes F_b^\vee$
$\Omega = \Omega_{\mathbb{P}/K}$	cotangent bundle on $\mathbb{P}$ over K
$\Omega^a = \bigwedge^a_{\mathcal{O}_{\mathbb{P}}} \Omega$	$\mathcal{O}_{\mathbb{P}}$-module of degree-a differential forms
$\mathcal{U} = \Omega(1)$	the tautological subbundle of rank $m-1$ in $\pi^* F$
$\mathcal{E} = \mathcal{U}^*$	the tautological quotient bundle of rank $m-1$ of $\pi^* F^\vee$
$\mathbb{K}_{>a}$, $\mathbb{K}_{\leqslant a}$	certain (shifted) truncations of $\mathbb{K}$
$\mathcal{M}^b_a(-c)$	$\mathscr{H}om_{\mathbb{P}}(\Omega^{b-1}(b), \Omega^{a-1}(a))(-c)$
$\mathcal{T}_a$ and $\mathcal{T}$	$p'^*(\Omega^{a-1}(a))$ and $\bigoplus_a \mathcal{T}_a$
M_a and M	$\operatorname{cok}(\bigwedge^a \varphi)$ and $\bigoplus_a M_a$
E	$\operatorname{End}_R(M)$
Q	Beĭlinson quiver on F
B	path algebra of Q
$\widetilde{\mathsf{Q}}$	doubled Beĭlinson quiver on F and G
C	quiverized Clifford algebra, path algebra of $\widetilde{\mathsf{Q}}$
Q^∞ and C^∞	infinite doubled Beĭlinson quiver and its path algebra
$\operatorname{Cliff}(b)$	Clifford algebra on the quadratic map b
$\mathfrak{rep}(\Gamma)$	abelian category of finite-dimensional representations of Γ
$\mathcal{R}$, $\widetilde{\mathcal{R}}$	certain representations of Q and $\widetilde{\mathsf{Q}}$
L_α	Schur module corresponding to the partition α

$$\begin{tikzcd}
\mathcal{Z} \arrow[dr, hook, "j"] \arrow[drr, bend left, "p'"] \arrow[dd, "q'"'] & & & \\
& \mathcal{Y} = H \times \mathbb{P} \arrow[r, "p"] \arrow[d, "q"'] & \mathbb{P} = \mathbb{P}(F^\vee) \arrow[r, "\pi"] & \operatorname{Spec} K \\
\operatorname{Spec} R \arrow[r, hook] & H = \operatorname{Hom}_K(G, F) & &
\end{tikzcd}$$

3. Direct Images of $\mathscr{H}om$ Between Bundles of Differential Forms

In Sections 3 through 5 we prove the technical results needed for the proofs of the theorems stated in the Introduction. At first reading the reader may wish to go directly to Section 6 (after a pit stop in §§5.1–5.2 to pick up the notation) where the applications start of the results obtained here.

The aim of the present section is to determine the higher direct images of the twisted bundles of homomorphisms between the modules of relative differential forms on a projective bundle. The result is surely not new; it contains, for example, Bott's formula for the twists of the differential forms themselves and the fact, first exploited by Beĭlinson in [1], that the direct sum $\bigoplus_i \Omega^i_{\mathbb{P}/K}(i)$ is a tilting bundle with its endomorphism ring isomorphic to a triangular version of the exterior algebra.

Not being aware of a complete, concise and explicit treatment of this general case in the literature, although it is certainly contained in the even more general treatment in [24], as well as to be able to use the ingredients of the proof later on, we recall here the argument that relies entirely on properties of the tautological Koszul complex, with the only challenge to keep the combinatorics at bay. To this end we first introduce compact notation we use throughout and then embark upon the actual computation after stating the result as Theorem 3.9.

3.1. *Notation.* We fix in this section a commutative base ring K and a projective K-module F of *constant finite rank* $m > 0$ in the sense that the K-module $\bigwedge^m_K F$ is invertible and faithful, equivalently, locally free of rank one.

The considerations to come will involve various multilinear operations on F and we choose abbreviated notation as follows.

- Unadorned tensor products are understood over K.
- $M^\vee$ denotes the K-dual of the K-module M.
- $F_a = \bigwedge^a_K F$ represents the indicated *exterior power* of F over K. It is a finite projective K-module, of constant rank $\binom{m}{a}$, non-zero for $0 \leqslant a \leqslant m$.

  The resulting abbreviation $F_a^\vee$ is unambiguous, as $\bigwedge^a(F^\vee) \cong (\bigwedge^a F)^\vee$ via a canonical and natural isomorphism of K-modules; see [16, XIX, Prop. 1.5].
- $F_a^b = \mathrm{Hom}_K(F_b, F_a) \cong F_a \otimes F_b^\vee$. The lower index thus indicates the covariant, the upper one the contravariant argument in the space of K-linear maps involved.
- $|F| = F_m$ denotes the *determinant* of F, a projective K-module of (constant) rank 1 by assumption. Again, $|F|^\vee \cong |F^\vee|$ canonically.
- $\mathbb{S}^b = \mathrm{Sym}^b_K(F)$ represents the indicated *symmetric power* of F over K. It is again a projective K-module, of constant rank $\binom{m-1+b}{b}$, non-zero for $b \geqslant 0$. We write $\mathbb{S} = \bigoplus_{b\geqslant 0} \mathbb{S}^b = \mathrm{Sym}_K F$ for the *symmetric algebra* on F over K, endowed with its canonical grading that places $\mathbb{S}^b$ into (internal) degree b.

Complexes will be graded *cohomologically*, so that the differential increments the complex degree by 1. Recall that the (simple) *translation* of a complex, denoted [1], then shifts a complex one place against the direction of the differential and changes the sign of said differential.

3.2. *The tautological Koszul complex.* Exterior and symmetric algebra on F over K combine to define the (affine) *tautological Koszul complex* $\mathbb{K}(\mathrm{id}_F)$ over the identity map on F; see [6, 9.3 AX.151]. That Koszul complex often plays a dominant role in (co-)homological considerations, and this instance is no exception.

Recall that the underlying bigraded $\mathrm{Sym}_K F$-module of $\mathbb{K}(\mathrm{id}_F)$ is $\bigwedge_K(F[1]) \otimes \mathrm{Sym}_K F$ and that the differential can be described in a coordinate-free manner through the comultiplication on the exterior algebra and the multiplication on the symmetric algebra. Namely, denote $\Delta^{a-1,1}\colon F_a \longrightarrow F_{a-1} \otimes F$ the indicated bihomogeneous component of the comultiplication defined by applying the exterior algebra functor to $\Delta\colon F \longrightarrow F \oplus F$ followed by the canonical isomorphism

$\bigwedge_K(F \oplus F) \cong \bigwedge_K F \otimes \bigwedge_K F$. With $\mu^{1,b}\colon F \otimes \mathbb{S}^b = \mathbb{S}^1 \otimes \mathbb{S}^b \longrightarrow \mathbb{S}^{b+1}$ the indicated bihomogeneous component of the multiplication on the symmetric algebra, the differential ∂ is then simply the direct sum of its bihomogeneous components

$$\partial_a^b\colon F_a \otimes \mathbb{S}^b \xrightarrow{\Delta^{a-1,1}\otimes \mathbb{S}^b} F_{a-1} \otimes F \otimes \mathbb{S}^b \xrightarrow{F_{a-1}\otimes \mu^{1,b}} F_{a-1} \otimes S^{b+1}\,.$$

We continually use the following basic fact.

Proposition 3.3 (cf. [6, 9.3 Prop. 3])**.** *The homogeneous strand of the Koszul complex $\mathbb{K}(\mathrm{id}_F)$ in internal degree $a \in \mathbb{Z}$ is a complex of finite projective K-modules of constant rank*

(3.3.1)
$$0 \longrightarrow |F| \otimes \mathbb{S}^{a-m} \longrightarrow F_{m-1} \otimes \mathbb{S}^{a-m+1} \longrightarrow \cdots \longrightarrow F \otimes \mathbb{S}^{a-1} \longrightarrow \mathbb{S}^a \longrightarrow 0\,.$$

It is supported on the integral interval $[-\min\{a,m\},0]$ and, unless $a = 0$, it is exact*, thus, even* split exact *as its terms are finite projective K-modules. If $a = 0$, the complex reduces to the single copy of $K \cong \mathbb{S}^0$ placed in (cohomological) degree 0.* □

3.4. *The projective tautological Koszul complex.* Now we turn to $\mathbb{P} = \mathbb{P}(F^\vee) = \mathrm{Proj}_K(\mathrm{Sym}_K F)$, the projective space of linear forms on F over K with structure morphism $\pi\colon \mathbb{P} \longrightarrow \mathrm{Spec}\, K$ and its canonical very ample line bundle $\mathcal{O}_{\mathbb{P}}(1)$. If M is any K-module, we write $M \otimes \mathcal{O}_{\mathbb{P}}$ for the induced $\mathcal{O}_{\mathbb{P}}$-module π^*M, and even $M(i) = M \otimes \mathcal{O}_{\mathbb{P}}(i)$ for any integer i.

The $\mathcal{O}_{\mathbb{P}}$-linear *Euler derivation* $e\colon F\otimes\mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}}$ corresponds to the identity on F under the standard identifications

$$\mathrm{Hom}_{\mathbb{P}}(F\otimes\mathcal{O}_{\mathbb{P}}(-1), \mathcal{O}_{\mathbb{P}}) \cong \mathrm{Hom}_{\mathbb{P}}(\pi^*F, \mathcal{O}_{\mathbb{P}}(1)) \cong \mathrm{Hom}_K(F, \pi_*\mathcal{O}_{\mathbb{P}}(1)) \cong \mathrm{Hom}_K(F,F)$$

It gives rise to the *(projective) tautological Koszul complex* of $\mathcal{O}_{\mathbb{P}}$-modules

$$\mathbb{K} \quad\equiv\quad 0 \longrightarrow F_m(-m) \longrightarrow \cdots \longrightarrow F(-1) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0\,, \tag{3.4.1}$$

where we place $\mathcal{O}_{\mathbb{P}}$ in (cohomological) degree zero, so that the complex is supported again on the interval $[-m,0]$. This complex on $\mathbb{P}$ is the sheafification of the affine tautological Koszul complex $\mathbb{K}(\mathrm{id}_F)$ in 3.2, and, conversely, if one applies π_* to $\mathbb{K}(a)$ for some integer a, the result is the homogeneous strand of the affine Koszul complex displayed in (3.3.1) above.

3.5. *Differential forms.* The Koszul complex $\mathbb{K}$ on $\mathbb{P}$ is exact and decomposes into short exact sequences

$$0 \longrightarrow \Omega^a \longrightarrow F_a(-a) \longrightarrow \Omega^{a-1} \longrightarrow 0 \tag{3.5.1}$$

where $\Omega^a = \bigwedge^a_{\mathbb{P}} \Omega^1_{\mathbb{P}/K}$ denotes the $\mathcal{O}_{\mathbb{P}}$-module of relative Kähler differential forms on $\mathbb{P}$ of degree a, equivalently, the locally free sheaf of sections of the a^{th} exterior power of the cotangent bundle on $\mathbb{P}$ relative to K.

Recall as well that the locally free $\mathcal{O}_{\mathbb{P}}$-module $\Omega^{m-1} \cong F_m(-m) = |F|(-m)$ of rank 1 represents $\omega_{\mathbb{P}/K}$, the *relative dualizing $\mathcal{O}_{\mathbb{P}}$-module* for the projective morphism π.

3.6. *The canonical (co-)resolutions of the differential forms.* Twisting, truncating, and translating the Koszul complex (3.4.1) appropriately provides locally free resolutions and coresolutions of the $\mathcal{O}_{\mathbb{P}}$-modules $\Omega^a(a')$, for any integers a, a'. These (co-)resolutions are represented by the following quasi-isomorphisms of complexes, where we view $\Omega^a(a')[0]$ as a complex concentrated in degree zero,

$$\begin{tikzcd} & \Omega^a(a')[0] \arrow[d, "i_a"', "\simeq"] \\ \big(0 \arrow[r] & F_a(a'-a) \arrow[r] & \cdots \arrow[r] & F(a'-1) \arrow[r] & \mathcal{O}_{\mathbb{P}}(a') \arrow[r] & 0\big)[-a] \end{tikzcd} \tag{3.6.1}$$

and

$$\begin{tikzcd} \big(0 \arrow[r] & {|F|}(a'-m) \arrow[r] & \cdots \arrow[r] & F_{a+1}(a'-a-1) \arrow[r] \arrow[d, "p_a"', "\simeq"] & 0\big)[-a-1] \\ & & & \Omega^a(a')[0] \end{tikzcd} \tag{3.6.2}$$

Denote $\mathbb{K}_{\leqslant a}(a')$ the locally free *coresolution* displayed in (3.6.1). It is thus the (cochain) complex concentrated on the interval $[0, a]$ with non-zero terms

$$\mathbb{K}_{\leqslant a}(a')^i = F_{a-i}(a' - a + i)$$

for $i = 0, \dots, a$, and with $\mathcal{H}^0(\mathbb{K}_{\leqslant a}(a')) \cong \Omega^a(a')$ the only possibly non-vanishing cohomology $\mathcal{O}_{\mathbb{P}}$-module.

Analogously the locally free *resolution* displayed in (3.6.2) is denoted $\mathbb{K}_{>a}(a')$. It is a (chain) complex concentrated on the interval $[a - m + 1, 0]$ with terms

$$\mathbb{K}_{>a}(a')^j = F_{a+1-j}(a' - a + j - 1)$$

for $j = a - m + 1, \dots, 0$, and with $\mathcal{H}^0(\mathbb{K}_{>a}(a')) \cong \Omega^a(a')$ the only possibly non-vanishing cohomology.

The proper signs of the differentials in these (co-)resolutions are uniquely determined by the requirement that the mapping cone over the composition

$$i_a p_a \colon \mathbb{K}_{>a}(a') \xrightarrow{p_a} \Omega^a(a')[0] \xrightarrow{i_a} \mathbb{K}_{\leqslant a}(a')$$

returns exactly $\mathbb{K}(a')[-a]$.

3.7. *The higher direct images.* The (co-)resolutions displayed in (3.6.1) and (3.6.2) combine to produce, for any integers a, b, and c, four ways to represent the locally free $\mathcal{O}_{\mathbb{P}}$-modules[1]

$$\begin{aligned} \mathcal{M}_a^b(-c) &= \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^{b-1}(b), \Omega^{a-1}(a))(-c) \\ &\cong \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^{b-1}(b-1), \Omega^{a-1}(a-1))(-c) \end{aligned}$$

as sole cohomology sheaf in total degree zero of a bicomplex[2] of locally free $\mathcal{O}_{\mathbb{P}}$-modules, each supported in exactly one of the four quadrants in the plane, representing a suitably twisted "cut-out" of the endomorphism complex $\mathrm{End}_{\mathcal{O}_{\mathbb{P}}}(\mathbb{K})$ of the projective tautological Koszul complex. Choosing the appropriate bicomplex, the total derived direct image of $\mathcal{M}_a^b(-c)$ can be obtained as the cohomology of just the (dual of the) direct image of that bicomplex.

The work is reduced considerably in view of the following.

[1]The curious looking notation will be justified later.

[2] bicomplex = total complex obtained from the corresponding (naive) double complex.

Lemma 3.8. *For any integers a, b, and c, there are canonical isomorphisms of locally free $\mathcal{O}_{\mathbb{P}}$-modules*

$$\mathcal{M}_a^b(-c) \cong \mathcal{M}_{m+1-b}^{m+1-a}(-c) \tag{3.8.1}$$

and

$$\mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{M}_a^b(-c), \Omega^{m-1}) \cong \mathcal{M}_b^a(c-m) \otimes_{\mathcal{O}_{\mathbb{P}}} \pi^*|F|\,. \tag{3.8.2}$$

Proof. The non-degenerate pairing resulting from exterior multiplication

$$- \wedge -\colon \Omega^{a-1}(a) \otimes_{\mathcal{O}_{\mathbb{P}}} \Omega^{m-a}(-a) \longrightarrow \Omega^{m-1} \tag{3.8.3}$$

induces for each integer a a natural isomorphism

$$\Omega^{m-a}(m+1-a) \xrightarrow{\cong} \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^{a-1}(a), \Omega^{m-1}(m+1)) \tag{3.8.4}$$

whence applying the contravariant functor $\mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(-, \Omega^{m-1}(m+1))$ to each argument returns an isomorphism

$$\begin{array}{c} \mathcal{M}_a^b(-c) = \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^{b-1}(b), \Omega^{a-1}(a))(-c) \\ \Big\downarrow{\scriptstyle\cong} \\ \mathcal{M}_{m+1-b}^{m+1-a}(-c) = \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^{m-a}(m+1-a), \Omega^{m-b}(m+1-b))(-c) \end{array}$$

as desired. Similarly one obtains from the definition of $\mathcal{M}_a^b(-c)$ and adjunction the first three isomorphisms in

$$\begin{aligned}
\mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{M}_a^b(-c), \Omega^{m-1}) &\cong \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^{b-1}(b), \Omega^{a-1}(a)), \Omega^{m-1})(c) \\
&\cong \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^{a-1}(a) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^{b-1}(b), \mathcal{O}_{\mathbb{P}}), \Omega^{m-1})(c) \\
&\cong \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^{a-1}(a), \Omega^{b-1}(b) \otimes_{\mathcal{O}_{\mathbb{P}}} \Omega^{m-1})(c) \\
&\cong \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^{a-1}(a), \Omega^{b-1}(b))(c-m) \otimes_{\mathcal{O}_{\mathbb{P}}} \pi^*|F| \\
&= \mathcal{M}_b^a(c-m) \otimes_{\mathcal{O}_{\mathbb{P}}} \pi^*|F|
\end{aligned}$$

while the fourth one uses the isomorphism $\Omega^{m-1} \cong |F|(-m)$ recalled in 3.5, and the final equality substitutes the definition of $\mathcal{M}_b^a$. $\square$

After these preliminary considerations we turn now to the determination of the higher direct images of $\mathcal{M}_a^b(-c)$ with respect to the projective morphism $\pi\colon \mathbb{P} \longrightarrow \operatorname{Spec} K$. The result is as follows, and the remainder of this section contains its detailed proof, followed by some immediate consequences.

Theorem 3.9. *For any integers a, b, c, and each $\nu \in \mathbb{Z}$, the higher direct image $\mathbf{R}^\nu\pi_*(\mathcal{M}_a^b(-c))$ of the locally free $\mathcal{O}_{\mathbb{P}}$-module*

$$\mathcal{M}_a^b(-c) = \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}\left(\Omega^{b-1}(b), \Omega^{a-1}(a)\right)(-c)$$

is a finite projective K-module. In particular, the higher direct images $\mathbf{R}^\nu\pi_*(\mathcal{M}_a^b(-c))$ are non-zero for at most one value of ν. In case $a+b \geqslant m+1$, the precise situation is as follows.

1. *For $c < 0$, only the direct image $\mathbf{R}^0\pi_*(\mathcal{M}_a^b(-c)) = \pi_*(\mathcal{M}_a^b(-c))$ itself is non-zero.*
2. *For $0 \leqslant c < a-b$, for $m-b < c < a$, and for $a+m-b < c \leqslant m$, all (higher) direct images vanish.*

(3) *For* $\max\{0, a-b\} \leqslant c \leqslant m-b$*, the only non-vanishing higher direct image is*

$$\mathbf{R}^c\pi_*\left(\mathcal{M}_a^b(-c)\right) \cong F^\vee_{c+b-a}\,.$$

(4) *For* $a \leqslant c \leqslant \min\{a+m-b, m\}$*, the only non-vanishing higher direct image is*

$$\mathbf{R}^{c-1}\pi_*(\mathcal{M}_a^b(-c)) \cong F^\vee_{c+b-a}\,.$$

(5) *For* $m < c$*, only the highest direct image is non-zero, and it satisfies*

$$\mathbf{R}^{m-1}\pi_*(\mathcal{M}_a^b(-c)) \cong \pi_*(\mathcal{M}_b^a(c-m))^\vee \otimes |F|^\vee\,.$$

The case $a+b < m+1$ *reduces to the previous one in light of Lemma 3.8.*

Remark 3.10. As the target of π is affine and each $\mathcal{M}_a^b(-c)$ is locally free, the usual local-global spectral sequence yields natural isomorphisms of K-modules

$$\mathbf{R}^\nu\pi_*(\mathcal{M}_a^b(-c)) \cong \operatorname{Ext}^\nu_{\mathcal{O}_\mathbb{P}}(\Omega^{b-1}(b'), \Omega^{a-1}(a'))$$

for any integers ν and a', b' with $b'-a' = c+b-a$.

On the other hand, as the calculation of higher direct images is local in the base, the reader may as well replace $\operatorname{Spec} K$ in Theorem 3.9 by an arbitrary scheme with a locally free sheaf F of constant rank m on it to obtain the analogous result for the higher direct images relative to a projective bundle over an arbitrary base scheme.

Remark 3.11. The results of the Theorem are invariant under the involution $(a,b,c) \leftrightarrow (m+1-b, m+1-a, c)$ in view of Lemma 3.8. Note further that either the first or the last interval in Theorem 3.9(2) is *empty*, depending on whether or not $a \leqslant b$.

Proof of Theorem 3.9. In view of the foregoing remark, we may assume without loss of generality that $a+b \geqslant m+1$. Using Grothendieck-Serre duality for the projective morphism π we show next that *it suffices to establish the claims for* $c \leqslant \frac{1}{2}(a-b+m)$.

Namely, assume we have shown that in the indicated range the higher direct images are finite projective K-modules and that at most one higher direct image $\mathbf{R}^\nu\pi_*(\mathcal{M}_a^b(-c))$ is not zero for given a, b, c. Using

$$\mathbf{R}^\nu\pi_*(\Omega^{m-1}) \cong \begin{cases} 0 & \text{for } \nu \neq m-1 \\ K & \text{for } \nu = m-1 \end{cases}$$

the duality theorem for projective morphisms yields that the natural pairing of complexes of K-modules

$$\mathbf{R}\pi_*(\mathcal{M}_a^b(-c)) \otimes^{\mathbb{L}} \mathbf{R}\pi_*(\mathscr{H}om_{\mathcal{O}_\mathbb{P}}(\mathcal{M}_a^b(-c), \Omega^{m-1})) \longrightarrow \mathbf{R}\pi_*(\Omega^{m-1}) \simeq K[1-m]$$

is non-degenerate. The isomorphism (3.8.2) together with the projection formula $\mathbf{R}\pi_*(- \otimes_{\mathcal{O}_\mathbb{P}} \pi^*|F|) \cong \mathbf{R}\pi_*(-) \otimes |F|$ let us rewrite this pairing as

$$\mathbf{R}\pi_*(\mathcal{M}_a^b(-c)) \otimes^{\mathbb{L}} \mathbf{R}\pi_*(\mathcal{M}_b^a(c-m)) \otimes^{\mathbb{L}} |F| \longrightarrow K[1-m]\,.$$

Accordingly, if the total direct image is represented by a single finite projective K-module in cohomological degree d, so that $\mathbf{R}\pi_*(\mathcal{M}_a^b(-c)) \simeq \mathbf{R}^d\pi_*\mathcal{M}_a^b(-c)[-d]$, we read off

$$\mathbf{R}^\nu\pi_*(\mathcal{M}_b^a(c-m)) \cong \begin{cases} 0 & \text{if } \nu \neq m-1-d \\ (\mathbf{R}^d\pi_*\mathcal{M}_a^b(-c))^\vee \otimes |F|^\vee & \text{if } \nu = m-1-d\,. \end{cases}$$

Under the involution $(a,b,c) \mapsto (a',b',c') = (b,a,m-c)$, the range

$$a+b \geqslant m+1 \quad , \quad \max\{0, a-b\} \leqslant c \leqslant m-b$$

is interchanged with the range

$$a'+b' \geqslant m+1 \quad , \quad a' \leqslant c' \leqslant \min\{a'-b'+m, m\}\,.$$

Now assuming that the conclusion of (3) holds, one finds on the one hand

$$\begin{aligned}(\mathbf{R}^c\pi_*(\mathcal{M}_a^b(-c))^\vee \otimes |F|^\vee &\cong \mathbf{R}^{m-1-c}\pi_*(\mathcal{M}_b^a(c-m))\\ &\cong \mathbf{R}^{c'-1}\pi_*(\mathcal{M}_{a'}^{b'}(-c'))\end{aligned}$$

and on the other

$$(F_{c+b-a}^\vee)^\vee \otimes |F|^\vee \cong F_{a'-b'+m-c'} \otimes |F|^\vee \cong F^\vee_{b'-a'+c'}$$

with the last isomorphism due to the pairing induced by exterior multiplication among the exterior powers of F.

In this way, (3) and (4) are seen to be dual statements. Similarly, the statements in (1) and (5) are dual to each other, while in (2) the statements for the first and third interval are interchanged, the statement for the middle one being selfdual.

It thus remains to prove the theorem for the range $a+b \geqslant m+1$ and $c \leqslant \frac{1}{2}(a-b+m)$. The outline of the argument here is as follows.

Depending on whether $c \leqslant m-b$ or $m-b < c$, we choose a different bicomplex to represent $\mathcal{M}_a^b(-c)$ in the derived category of $\mathbb{P}$. The choice is made so that the individual terms of the representing bicomplex are π_*-*acyclic*, that is, the direct image itself will be the only non-vanishing (higher) direct image, and the bicomplex resulting from applying π_* will represent $\mathbf{R}\pi_*(\mathcal{M}_a^b(-c))$ as a complex of finite projective K-modules. We then analyse this bicomplex along its "rows". Each of these is the tensor product over K of a finite projective K-module with a (subcomplex of a) homogeneous strand of the affine tautological Koszul complex $\mathbb{K}(\mathrm{id}_F)$, thus, is a complex with easily determined cohomology. It then remains to assemble the information so gained.

Now we turn to the details, where we freely use the well known results on the higher direct images of the locally free $\mathcal{O}_\mathbb{P}$-modules $\mathcal{O}_\mathbb{P}(i), i \in \mathbb{Z}$; see [12, Prop. 2.1.12] for the general case treated here.

3.12. With $a+b \geqslant m+1$, assume first $c \leqslant m-b$. Choosing for each of $\Omega^{a-1}(a)$ and $\Omega^{b-1}(b)$ the appropriate coresolution (3.6.1), one can represent $\mathcal{M}_a^b(-c)$ by the following bicomplex with non-zero terms concentrated in the fourth quadrant:

$$\begin{aligned}\mathbb{E}_{+-}^{i,-j} &= \mathscr{H}om_{\mathcal{O}_\mathbb{P}}(\mathbb{K}_{\leqslant b-1}(b), \mathbb{K}_{\leqslant a-1}(a))^{i,-j}(-c)\\ &\cong F_{a-1-i}(i+1) \otimes F^\vee_{b-1-j}(-1-j)(-c)\\ &\cong F_{a-1-i}^{b-1-j}(i-j-c) \quad \text{for } i,j \geqslant 0\,.\end{aligned}$$

This bicomplex evidently has the following properties:

(a) Each term $\mathbb{E}_{+-}^{i,-j}$ is the twist by a power of the distinguished very ample line bundle on $\mathbb{P}$ of an $\mathcal{O}_\mathbb{P}$-module induced from a finite projective K-module;
(b) The twist occurring in $\mathbb{E}_{+-}^{i,-j}$ depends only upon the total degree $i-j$;
(c) The bicomplex is supported on the rectangle $[0, a-1] \times [-b+1, 0]$ in the (i,j)-plane.

(d) The twists occurring in non-zero terms range over the integers from $1-b-c$ to $a-1-c$, an integral interval of length $a+b-2$.

(e) As $c \leqslant m-b$ and $a+b \geqslant m+1$ by assumption, the possible twists ?(t) occurring in non-zero terms of $\mathbb{E}_{+-}^{i,-j}$ satisfy $t \geqslant 1-m$, whence for each such term the higher direct images *vanish*, $\mathbf{R}^{\nu}\pi_*(\mathbb{E}_{+-}^{i,-j}) = 0$ for $\nu \neq 0$.

Property (e) implies in particular that the total higher direct image of $\mathcal{M}_a^b(-c)$ is represented in the derived category of K by π_* of this bicomplex,

$$\mathbf{R}\pi_*(\mathcal{M}_a^b(-c)) \simeq \pi_*\mathbb{E}_{+-}^{\bullet,\bullet}\,.$$

3.13. The cohomology of $\pi_*\mathbb{E}_{+-}^{\bullet,\bullet}$ is now readily determined by looking first at the "rows" of the bicomplex. Fixing $j \in [0, b-1]$, the corresponding (row) complex $\pi_*\mathbb{E}_{+-}^{\bullet,-j}$ is concentrated on the line segment $[0, a-1] \times \{-j\}$ in the (i, j)-plane and has the form

(3.13.1)
$$(0 \longrightarrow F_{a-1} \otimes \mathbb{S}^{-j-c} \longrightarrow F_{a-2} \otimes \mathbb{S}^{1-j-c} \longrightarrow \cdots \longrightarrow \mathbb{S}^{a-1-j-c} \longrightarrow 0) \otimes F_{b-1-j}^{\vee}[j]\,.$$

This complex is, up to the signs of the differentials, the translation by $[j]$ of the tensor product over K of $F_{b-1-j}^{\vee}$ with a subcomplex of the homogeneous strand in internal degree $a-1-j-c$ in the affine tautological Koszul complex $\mathbb{K}(\mathrm{id}_F)$ recalled in 3.4. That strand of $\mathbb{K}(\mathrm{id}_F)$ is exact except possibly at its ends. More precisely, the situation is as follows.

Lemma 3.14. *For* $(i, j) \in [0, a-1] \times [0, b-1]$ *and* $c \leqslant m-b$*, the cohomology* $H^{i,-j}(\pi_*\mathbb{E}_{+-}^{\bullet,-j})$ *of the row complex just displayed in (3.13.1) is non-zero only if*

(1) $(i, -j) = (0, -j)$ *with* $-j > c$*, and then*

$$\begin{aligned} H^{0,-j}(\pi_*\mathbb{E}_{+-}^{\bullet,-j}) &\cong \pi_*(\Omega^{a-1}(-j-c-1)) \otimes F_{b-1-j}^{\vee} \\ &\cong \ker\left(F_{a-1} \otimes \mathbb{S}^{-j-c} \longrightarrow F_{a-2} \otimes \mathbb{S}^{1-j-c}\right) \otimes F_{b-1-j}^{\vee} \\ &\cong \operatorname{cok}\left(F_{a+1} \otimes \mathbb{S}^{-2-j-c} \longrightarrow F_a \otimes \mathbb{S}^{-1-j-c}\right) \otimes F_{b-1-j}^{\vee} \end{aligned}$$

or

(2) $(i, -j) = (a-1, c-a+1)$*, in which case*

$$H^{a-1,c-a+1}(\pi_*\mathbb{E}_{+-}^{\bullet,-j}) \cong F_{b-1-j}^{\vee} \cong F_{c+b-a}^{\vee}\,.$$

In either case, the cohomology is a finite projective K*-module.* □

3.15. *Visualization.* The reader might find it helpful to contemplate the following visualisations in the (i, j)-plane, where the place with non-zero homology with respect to the horizontal differential is marked $\times$, those places with $\pi_*\mathbb{E}_{+-}^{i,j} \neq 0$ but no horizontal homology are marked by $\bullet$, and the symbol $\circ$ refers to entries where $\pi_*\mathbb{E}_{+-}^{i,j}$ is zero.

We begin with the simplest case, when $0 \leqslant c \leqslant m - b$, whence in particular $0 \leqslant c \leqslant a - 1$. We get then the following picture:

$$\pi_*\mathbb{E}_{+-}^{\bullet,\bullet} \quad \equiv \quad \begin{array}{|c|cccccccc|}
\hline
0 & \circ & \cdots & \circ & \bullet & \bullet & \cdots & \bullet & \bullet \\
 & \circ & \cdots & \circ & \circ & \bullet & \cdots & \bullet & \bullet \\
 & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & \circ & \cdots & \circ & \circ & \circ & \cdots & \bullet & \bullet \\
c-a+1 & \circ & \cdots & \circ & \circ & \circ & \cdots & \circ & \times \\
 & \circ & \cdots & \circ & \circ & \circ & \cdots & \circ & \circ \\
 & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1-b & \circ & \cdots & \circ & \circ & \circ & \cdots & \circ & \circ \\
\hline
 & 0 & & & c & & & & a-1 \\
\hline
\end{array} \tag{3.15.1}$$

In other words, there is at most one non-vanishing cohomology group occurring in those rows, whence the entire bicomplex equally only carries this cohomology. Note that that cohomology indeed appears if, and only if, $\max\{0, a - b\} \leqslant c$.

In summary, we read off the following result that settles the claims in Theorem 3.9 for c in the interval $[0, m - b]$.

Proposition 3.16. *For $a + b \geqslant m + 1$ and $0 \leqslant c < a - b$, the higher direct images of $\mathcal{M}_a^b(-c)$ all vanish, while for $\max\{0, a - b\} \leqslant c \leqslant m - b$ the only non-vanishing one is the finite projective K-module*

$$\mathbf{R}^c\pi_*(\mathcal{M}_a^b(-c)) \cong F_{c+b-a}^{\vee}\,.$$

□

3.17. In case $c < 0$, the corresponding diagram has the form

$$\pi_*\mathbb{E}_{+-}^{\bullet,\bullet} \quad \equiv \quad \begin{array}{|c|ccccccc|}
\hline
0 & \times & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \\
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
c & \times & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \\
 & \circ & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \\
 & \circ & \circ & \bullet & \cdots & \bullet & \bullet & \bullet \\
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 & \circ & \circ & \circ & \cdots & \bullet & \bullet & \bullet \\
 & \circ & \circ & \circ & \cdots & \circ & \bullet & \bullet \\
c-a+1 & \circ & \circ & \circ & \cdots & \circ & \circ & \times \\
 & \circ & \circ & \circ & \cdots & \circ & \circ & \circ \\
 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1-b & \circ & \circ & \circ & \cdots & \circ & \circ & \circ \\
\hline
 & 0 & & & & & & a-1 \\
\hline
\end{array}$$

with non-zero cohomology along the rows thus occurring only for total degrees in the interval $[\max\{1 - b, c\}, 0]$.

Now $\mathbf{R}^\nu\mathcal{M} = 0$ for $\nu < 0$ and any $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{M}$, whence the bicomplex $\pi_*\mathbb{E}_{+-}^{\bullet,\bullet}$ that represents $\mathbf{R}\pi_*(\mathcal{M}_a^b(-c))$ only admits cohomology in non-negative degrees. Combining these two facts, there can be at most a single degree, namely 0, in which there is non-vanishing cohomology. This amounts to the following result.

Proposition 3.18. *For $a+b \geqslant m+1$ and $c<0$, the higher direct images of $\mathcal{M}_a^b(-c)$ vanish except possibly*[3] *for the direct image $\pi_*(\mathcal{M}_a^b(-c))$ itself, a finite projective K-module of constant rank.*

Proof. We already explained before stating the proposition why the higher direct images necessarily vanish in degrees different from zero. The final statement follows then from the universality of the construction: the determination of the higher direct images of $\mathcal{M}_a^b(-c)$ as described is compatible with base change in the base $\operatorname{Spec} K$, and the fact that the higher direct images are concentrated in degree zero is independent of that base. It follows that $\pi_*(\mathcal{M}_a^b(-c))$ is K-flat, thus finite projective over K as it is finitely presented. Its rank can be computed through the Euler characteristic of the ranks of the terms of the bicomplex, whence the result is still constant across $\operatorname{Spec} K$. ☐

At this stage, we have established the claims in Theorem 3.9 for $a+b \geqslant m+1$ and $c \leqslant m-b$.

3.19. For further use we give in Lemma 3.20 below a concrete interpretation of the isomorphism

$$\mathbf{R}^c\pi_*(\mathcal{M}_a^b(-c)) \cong F_{c+b-a}^\vee\,. \tag{3.19.1}$$

for $a+b \geqslant m-1$ and $0 \leqslant c \leqslant m-b$ (see Proposition 3.16).

It will be convenient to regard $\mathbb{K} = \bigwedge(F(-1)[1])$ as a $\mathcal{O}_{\mathbb{P}}$-linear differentially graded algebra with differential d obtained by extending the Euler map $F(-1) \longrightarrow \mathcal{O}_{\mathbb{P}}$. For $u,v \in \mathbb{Z}$ we regard $\mathbb{K}(u)[v]$ as a $\mathcal{O}_{\mathbb{P}}$-linear $\mathbb{K}$-DG-bimodule.

Let $\lambda \in F^\vee$. By extending the linear map $\lambda\colon \mathbb{K}_1 = F(-1) \longrightarrow \mathcal{O}_{\mathbb{P}}(-1) = \mathbb{K}(-1)_0$ we obtain a derivation $\mathbb{K} \longrightarrow \mathbb{K}(-1)[1]$ which we denote by ∂_λ. The commutator $d\partial_\lambda + \partial_\lambda d$ is a derivation and since it is zero on generators it follows $d\partial_\lambda + \partial_\lambda d = 0$. A similar argument shows $\partial_\lambda\partial_{\lambda'} + \partial_{\lambda'}\partial_\lambda = 0$.

If $\lambda^1 \wedge \cdots \wedge \lambda^p \in F_p^\vee$ then we obtain a corresponding differential operator $\partial_{\lambda^1}\cdots\partial_{\lambda^p} : \mathbb{K} \longrightarrow \mathbb{K}(-p)[p]$ commuting with d. This yields a map of complexes

$$F_p^\vee \otimes \mathbb{K} \longrightarrow \mathbb{K}(-p)[p]\,.$$

Put $p = b+c-a$. We obtain a map

$$F_{b+c-a}^\vee \otimes \mathbb{K}(b)[-b+1] \longrightarrow \mathbb{K}(a-c)[c-a+1]\,.$$

Truncating in homological degree 0 and taking into account the shift incorporated into the definition of $\mathbb{K}_{\leqslant b+1}$ (see §3.6) we obtain a map

$$F_{b+c-a}^\vee \otimes \mathbb{K}_{\leqslant b-1}(b) \longrightarrow \mathbb{K}_{\leqslant -1+a-c}(a-c) \subset \mathbb{K}_{\leqslant a-1}(a)(-c)[c]\,. \tag{3.19.2}$$

Lemma 3.20. *The map*

$$F_{b+c-a}^\vee \longrightarrow \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathbb{K}_{\leqslant b-1}(b), \mathbb{K}_{\leqslant a-1}(a)(-c))[c] = \mathbf{R}\pi_*(\mathcal{M}_a^b(-c))[c] \tag{3.20.1}$$

obtained from (3.19.2) *is a quasi-isomorphism.*

Proof. Filtering the double complex $\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathbb{K}_{\leqslant b-1}(b), \mathbb{K}_{\leqslant a-1}(a))(-c)$ by rows as before, it is sufficient to show that the induced map to the only row carrying non-trivial cohomology is a quasi-isomorphism. Looking at the picture (3.15.1) we see

[3] We will see below in Remark 5.4 that they are indeed non-zero!

that we have to show that

$$\begin{aligned} F^\vee_{b+c-a} &\longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathbb{K}_{\leqslant b-1}(b)^{a-c-1}, \mathbb{K}_{\leqslant a-1}(a)^{a-1})(-c) \\ &= \pi_*(F^\vee_{b-1-(a-c-1)}(-1-(a-c-1)) \otimes \mathcal{O}_{\mathbb{P}}(1+(a-1))(-c)) \\ &= \pi_*(F^\vee_{b+c-a} \otimes \mathcal{O}_{\mathbb{P}}) \\ &= F^\vee_{b+c-a} \end{aligned}$$

is an isomorphism. This is an easy verification. □

3.21. Now we turn to the higher direct images of $\mathcal{M}^b_a(-c)$ in the range $m-b < c < a \leqslant m$, still under the assumption that $a+b \geqslant m+1$. To this end, we choose the locally free coresolution for $\Omega^{a-1}(a)$ as in (3.6.1), but the locally free *resolution* for $\Omega^{b-1}(b)$ as in (3.6.2), to represent $\mathcal{M}^b_a(-c)$ by the resulting bicomplex concentrated in the first quadrant. It has the terms

$$\begin{aligned} \mathbb{E}^{i,j}_{++} &= \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathbb{K}_{>b-1}(b), \mathbb{K}_{\leqslant a-1}(a))^{i,j}(-c) \\ &\cong F_{a-1-i}(i+1) \otimes F^\vee_{b+j}(j)(-c) \\ &\cong F^{b+j}_{a-1-i}(i+j+1-c) \quad \text{for } i,j \geqslant 0\,. \end{aligned}$$

Take note of the following properties, analogous to the properties (a) through (e) in §3.12 above.

(a) Each term $\mathbb{E}^{i,j}_{++}$ is the twist of a power of the canonical line bundle on $\mathbb{P}$ with an $\mathcal{O}_{\mathbb{P}}$-module induced from a finite projective K-module;
(b) The twist occurring in $\mathbb{E}^{i,j}_{++}$ only depends upon the total degree $i+j$;
(c) The bicomplex is supported on the rectangle $[0, a-1] \times [0, m-b]$ in the (i,j)-plane.
(d) The twists occurring in non-zero terms range over the integers from $1-c$ to $a-b-c+m$, an integral interval of length $a-b+m-1$.
(e) In view of the preceding point, and as $0 \leqslant m-b < c \leqslant m$ by our current assumption, the possible twists $?(t)$ occurring in non-zero terms of $\mathbb{E}^{i,j}_{++}$ satisfy $a > t \geqslant 1-m$ and the lower bound shows again that for each such term the higher direct images *vanish*, $\mathbf{R}^\nu\pi_*(\mathbb{E}^{i,j}_{++}) = 0$ for $\nu \neq 0$.

As before, property (e) implies in particular that the total derived direct image of $\mathcal{M}^b_a(-c)$ is represented by the direct image under π_* of this bicomplex,

$$\mathbf{R}\pi_*\mathcal{M}^b_a(-c) \simeq \pi_*\mathbb{E}^{\bullet,\bullet}_{++}$$

and we will determine its cohomology once again by looking first at the corresponding "row" complexes. Fixing therefore $j \in [0, m-b]$, the complex $\pi_*\mathbb{E}^{\bullet,j}_{++}$ is concentrated on the line segment $[c-j-1, a-1] \times \{j\}$ in the (i,j)-plane and has the form

$$(0 \longrightarrow F_{a+j-c} \longrightarrow F_{a+j-c-1} \otimes \mathbb{S}^1 \longrightarrow \cdots \longrightarrow \mathbb{S}^{a+j-c} \longrightarrow 0) \otimes F^\vee_{b+j}[-j]\,.$$

Up to the signs of the differentials, this is the translation by $[-j]$ of the tensor product over K of $F^\vee_{b+j}$ with the (entire!) homogeneous strand in internal degree $a+j-c$ in the affine tautological Koszul complex $\mathbb{K}(\mathrm{id}_F)$ recalled in 3.4.

Note that $a-m \leqslant a+j-c \leqslant a-1$ by the assumptions $c \in [m-b+1, m]$ and $j \in [0, m-b]$, whence either

(i) $a+j-c < 0$, and this complex has *no non-zero terms*, or

(ii) $a+j-c=0$, and the strand of the affine Koszul complex has cohomology equal to its only non-zero term, isomorphic to K, in bidegree $(a-1, c-a)$, thus total degree $c-1$, or

(iii) $0<a+j-c \leqslant a-1$, and the strand of the Koszul complex is *exact*.

Depicting the situation again, for $m-b<c<a$ the resulting diagram is of the form

$$\pi_*\mathbb{E}^{\bullet,\bullet}_{++} \quad \equiv \quad \begin{array}{|c|cccccccccc|}
\hline
m-b & \circ & \cdots & \circ & \bullet & \bullet & \cdots & \bullet & \cdots & \bullet \\
 & \circ & \cdots & \circ & \circ & \bullet & \cdots & \bullet & \cdots & \bullet \\
 & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \circ & \cdots & \circ & \circ & \circ & \cdots & \bullet & \cdots & \bullet \\
\hline
 & 0 & & & c' & & & & & a-1 \\
\hline
\end{array}$$

where we have set $c'=c-(m-b+1)$, satisfying $0 \leqslant c' < a+b-m-1 \leqslant a-1$. In other words, all the rows here are already exact, so there are *no non-zero higher direct images*. We record this as the following result that covers the claims in Theorem 3.9 for the range $m-b<c<a$.

Proposition 3.22. *For $a+b \geqslant m+1$ and $m-b<c<a$, all higher direct images of $\mathcal{M}^b_a(-c)$ vanish, $\mathbf{R}^\nu\pi_*(\mathcal{M}^b_a(-c))=0$ for each integer ν.* □

The proof of Theorem 3.9 is now complete, in view of the duality considerations at the beginning. □

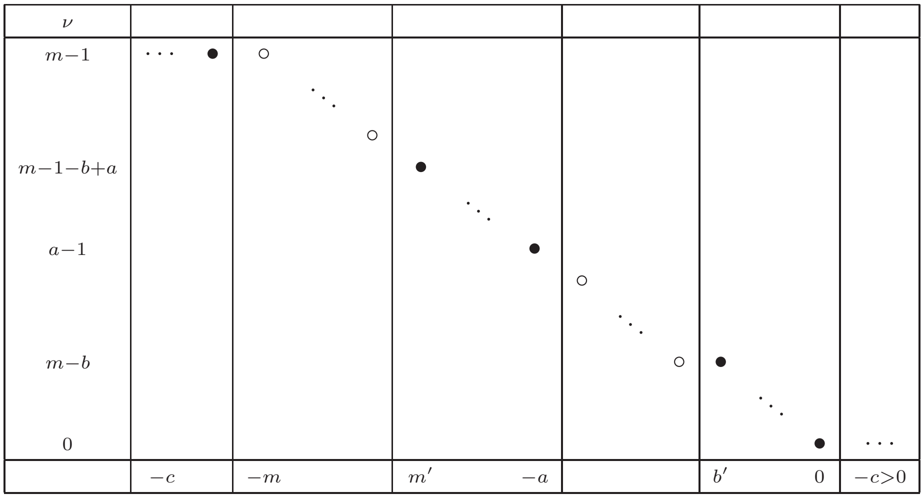
For the benefit of the reader and for further use below, we visualize the results in Theorem 3.9 as follows.

3.23. For $a+b \geqslant m+1$ and $m \geqslant a \geqslant b \geqslant 1$, depicting by $\bullet$ the nonzero terms $\mathbf{R}^\nu\pi_*(\mathcal{M}^b_a(-c))$, by $\circ$ or just empty spaces the vanishing ones, and setting $m'=b-m, a'=b-a$ for formatting purposes, results in the following diagram in the $(-c,\nu)$-plane

ν						
$m-1$	$\cdots$	$\bullet$				
		$\ddots$				
$a-1$		$\bullet$	$\circ$			
			$\ddots$			
			$\circ$			
$m-b$				$\bullet$		
				$\ddots$		
$a-b$				$\bullet$		
					$\circ$	
					$\ddots$	
0					$\circ$	$\bullet$ $\cdots$
	$-c$	$-m$ $\quad$ $-a$		m' $\quad$ a'	0	$-c>0$

while the corresponding diagram for $a+b \geqslant m+1$ and $m \geqslant b \geqslant a \geqslant 1$ is obtained from the above through a halfturn. It looks as follows, where we have set this time

$m' = b - a - m$ and $b' = b - m$,



Theorem 3.9 can also be reformulated in the following terms, which are the most useful for the application we have in mind.

Corollary 3.24. *With notations as in 3.9, for arbitrary integers a, b, c, d, the higher direct image $\mathbf{R}^d\pi_*(\mathcal{M}_a^b(-c))$ is* not zero *only in the following cases:*

(1) *If $d - c > 0$, then $d = 0$ and, necessarily, $c < 0$.*
(2) *If $d - c = 0$, then $c + b \in [\max\{a, b\}, \min\{m, a + b - 1\}]$.*
(3) *If $d - c = -1$, then $c - a \in [\max\{0, m - a - b - 1\}, \min\{m - b, m - a\}]$.*
(4) *If $d - c < -1$, then $d = m - 1$ and, necessarily, $c > m$.*

Using Remark 3.10, for $c = 0$, Theorem 3.9 returns the following well known fact, namely that the sequence $\mathcal{O}_{\mathbb{P}}(-1) \cong \Omega^{m-1}(m-1), \dots, \Omega^0(0) = \mathcal{O}_{\mathbb{P}}$ of locally free $\mathcal{O}_{\mathbb{P}}$-modules "between $\mathcal{O}_{\mathbb{P}}(-1)$ and $\mathcal{O}_{\mathbb{P}}$" is *strongly exceptional* in the sense of Bondal [4].

Corollary 3.25. *In case $c = 0$, we have $\mathcal{M}_{a+1}^{b+1} \cong \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^b(b), \Omega^a(a))$ and 3.9 yields*

$$\operatorname{Ext}^i_{\mathcal{O}_{\mathbb{P}}}(\Omega^b(b), \Omega^a(a)) \cong \begin{cases} \bigwedge_K^{b-a} F^\vee & \text{if } i = 0 \text{ and } b \geqslant a, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

To end this section, we determine the ranks of the finite projective K-modules occurring in Theorem 3.9. They can be easily determined in closed form by means of the Hilbert-Serre, a.k.a. the Hirzebruch-Riemann-Roch Theorem and the result is as follows.

Corollary 3.26. *Let $m \geqslant a, b \geqslant 1$ be integers with $a + b \geqslant m + 1$. Denote $r_a^b(z) \in \mathbb{Q}[z]$ the unique polynomial of degree at most $m - 1$ that at the integers in* *the interval* $[-m,0]$ *takes on the values*

$$r_a^b(-c)=\begin{cases}0 & \text{for } 0\leqslant c<a-b,\\ (-1)^c\binom{m}{c+b-a} & \text{for } \max\{0,a-b\}\leqslant c\leqslant m-b,\\ 0 & \text{for } m-b<c<a,\\ (-1)^{c-1}\binom{m}{c+b-a} & \text{for } a\leqslant c\leqslant \min\{a+m-b,m\},\\ 0 & \text{for } a+m-b<c\leqslant m.\end{cases}$$

The ranks of the higher direct images of $\mathcal{M}_a^b(-c)$ *are then determined uniquely through*

$$\sum_\nu(-1)^\nu \operatorname{rank}_K \mathbf{R}^\nu\pi_*(\mathcal{M}_a^b(-c))=r_a^b(c)$$

as for each triple a,b,c *at most one term in the sum on the left is nonzero.*

4. Interlude: Projective Resolutions from Sparse Spectral Sequences

In this section we record a "degeneracy result" that allows one to obtain a projective resolution of a bicomplex from one of the associated spectral sequences, provided the corresponding first page is "sparse" with projective terms. The result applies to bicomplexes in any abelian category A with enough projectives, whence we assume here this setting.

4.1. *Categorical notation.* Let $\mathsf{K}^-=\mathsf{K}^-(\mathbb{P}\mathsf{A})$ denote the homotopy category of complexes of projectives from A that are bounded in the direction of the differential, and, for an arbitrary complex C over A, denote by K^-/C the corresponding *comma category*; see [18, II.6]. Its objects are thus homotopy classes of morphisms of complexes $\varphi\colon P\longrightarrow C$ with $P\in\mathsf{K}^-$, and its morphisms from $\varphi\colon P\longrightarrow C$ to $\varphi'\colon P'\longrightarrow C$ are those homotopy classes of morphisms of complexes $\psi\colon P\longrightarrow P'$, for which $\varphi'\psi=\varphi$ in K^-.

Recall that a morphism of complexes is a *quasi-isomorphism* if it induces an isomorphism in cohomology. If C is any complex over A, then a *projective resolution* of C is any quasi-isomorphism $\varphi\colon P\longrightarrow C$ with source in K^-. Such a projective resolution, if it exists, is an object in K^-/C, and in there it is unique up to isomorphism.

4.2. *Assumptions.* Fix henceforth a bicomplex $C=(C^{i,j},d)$ supported on the *upper half-plane* $(i,j)\in\mathbb{Z}\times\mathbb{N}$ and whose associated total (direct sum) complex exists in the given abelian category A. Equivalently, the (countable) direct sums $C^n=\bigoplus_{i+j=n}C^{i,j}$ exist in A for each integer n. As C is a bicomplex, the differential of C decomposes as $d=d_h+d_v$, where

$$d_h=\bigoplus_{i,j}d_h^{i,j}\quad,\quad d_h^{i,j}\colon C^{i,j}\longrightarrow C^{i+1,j}$$

$$d_v=\bigoplus_{i,j}d_v^{i,j}\quad,\quad d_v^{i,j}\colon C^{i,j}\longrightarrow C^{i,j+1}$$

represent, respectively, the horizontal and vertical components.

Filtering the bicomplex according to column degree, the resulting spectral sequence *converges* against the cohomology of C, as the bicomplex is supported on the upper half-plane, and it displays on its first page the vertical cohomology groups. In short,

$$E_1^{i,j} = H_v^{i,j}(C) = H^{i,j}(C, d_v) \quad \Longrightarrow \quad H^{i+j}(C)\,.$$

Below we will use the following basic fact.

Lemma 4.3. *Let $D = (D, d)$ be a complex whose cohomology objects $H^n(D)$ are* projective *in A for each integer n. Viewing the graded object $H = \bigoplus_n H^n(D)[-n]$ over A as a complex with zero differential, there exists a quasi-isomorphism from it to D. In other words, the cohomology itself constitutes a projective resolution of D.*

Proof. Indeed, let Z denote the complex of cycles, which sits naturally as a subcomplex of D with zero differential. The natural epimorphism $Z \twoheadrightarrow H$ of graded objects, or complexes with zero differentials, admits a section $H \hookrightarrow Z$, as the components of H are projective. The resulting composition $H \hookrightarrow Z \hookrightarrow D$ provides for the desired quasi-isomorphism. □

Now we can formulate the "degeneracy criterion".

Proposition 4.4. *With C as in 4.2, suppose that each of its vertical cohomology groups $E_1^{i,j}$ is* projective *and assume further that there exist an integer a and a strictly decreasing sequence of integers $i_a > i_{a-1} > \cdots$ such that $E_1^{i,j} = 0$ for*

- *$i > i_a$ and all j, and for*
- *$i + j \neq n$ when $i_{n-1} < i \leqslant i_n$.*

In this case,

1. *for each integer n, the direct sum $P^n = \bigoplus_{i+j=n} E_1^{i,j}$ exists and is* projective *in A; note that $P^n = 0$ if $n > a$;*
2. *there exist morphisms $\{\partial^n \colon P^n \longrightarrow P^{n+1}\}_{n\in\mathbb{Z}}$ with $\partial^{n+1}\partial^n = 0$, whence $P = (P^n, \partial^n)$ constitutes a complex in K^-;*
3. *there exists a quasi-isomorphism $\varphi = \{\varphi^n\}_{n\in\mathbb{Z}} \colon P \longrightarrow C$.*

In particular, $\varphi\colon P \longrightarrow C$ constitutes a projective resolution of C.

Proof. Consider the (naive) ascending and exhaustive filtration

$$F_{a+1} = (C^{i,j})_{i>i_a,j} \hookrightarrow \cdots \hookrightarrow F_n = (C^{i,j})_{i>i_{n-1},j} \hookrightarrow \cdots \hookrightarrow C$$

on the bicomplex C. Each bicomplex F_ν, for $\nu \leqslant a+1$, satisfies the same hypotheses as those assumed for C, and we first establish the theorem for these bicomplexes by descending induction. The proof will then be finished by passing to the limit.

The bicomplex F_{a+1} is *exact*: indeed, its vertical cohomology $E_1^{i,j}(F_{a+1})$ vanishes by assumption, whence the (total) cohomology of F_{a+1} is equally 0 as the associated spectral sequence, essentially concentrated in the first quadrant, converges. Accordingly, for $a' \geqslant a$, we get $P^{a'} = \bigoplus_{i+j=a'} E_1^{i,j} = 0$, and so $\partial^{a'} = 0$, with $\varphi_{a+1}\colon 0 \longrightarrow F_{a+1}$ a quasi-isomorphism. This establishes the initial step of the induction.

Now assume that for some integer $\nu \leqslant a$,

(i) the terms $P^{\nu'} = \bigoplus_{i+j=\nu'} E_1^{i,j}$ exist and are projective in A for $\nu' > \nu$,
(ii) we have constructed a complex

$$\mathbb{P}^{\nu+1} \equiv \ (0 \longrightarrow P^{\nu+1} \xrightarrow{\partial^{\nu+1}} P^{\nu+2} \longrightarrow \cdots \longrightarrow P^a \xrightarrow{\partial^a} 0)$$

(iii) and a quasi-isomorphism $\varphi_{\nu+1}\colon \mathbb{P}^{\nu+1} \longrightarrow F_{\nu+1}$.

By definition of the filtration, the quotient $F_\nu/F_{\nu+1}$ is a bicomplex concentrated on the vertical strip $[i_{\nu-1}+1, i_\nu] \times \mathbb{N}$. On this strip of finite width, the vertical cohomology is by hypothesis concentrated in total degree ν, and so involves only finitely many terms. Accordingly, $P^\nu = \bigoplus_{i+j=\nu} E_1^{i,j} \cong \bigoplus_{i=i_{\nu-1}+1}^{i_\nu} E_1^{i,\nu-i}$ is a finite direct sum of projectives, thus, exists and is itself projective in A.

Moreover, $P^\nu[-\nu]$ represents the (total) cohomology of $F_\nu/F_{\nu+1}$, as the spectral sequence $E_1^{i,j}(F_\nu/F_{\nu+1}) \implies H^{i+j}(F_\nu/F_{\nu+1})$ collapses on its first page, due to the lack of cohomology outside the diagonal $i+j=\nu$. As P^ν is projective, it follows from Lemma 4.3 that there exists then a quasi-isomorphism of complexes $\chi^\nu\colon P^\nu[-\nu] \xrightarrow{\cong} F_\nu/F_{\nu+1}$ from the complex with P^ν as sole possibly non-zero term in degree ν to $F_\nu/F_{\nu+1}$; it constitutes a projective resolution of $F_\nu/F_{\nu+1}$.

The semi-split exact sequence of complexes

$$0 \longrightarrow F_{\nu+1} \longrightarrow F_\nu \longrightarrow F_\nu/F_{\nu+1} \longrightarrow 0$$

defines an exact triangle in the derived category $\mathcal{D}(\mathsf{A})$ that together with the already constructed quasi-isomorphisms accounts for the solid arrows in

$$\begin{tikzcd}
\mathbb{P}^{\nu+1} \arrow[r, dotted] \arrow[d, "\varphi^{\nu+1}"', "\simeq"] & \mathsf{cone}(\delta^\nu) \arrow[r, dotted] \arrow[d, dotted, "\varphi^\nu"', "\simeq"] & P^\nu[-\nu] \arrow[r, dashed, "\delta^\nu"] \arrow[d, "\simeq"', "\chi^\nu"] & \mathbb{P}^{\nu+1}[1] \arrow[d, "\simeq"', "\varphi^{\nu+1}[1]"] \\
F_{\nu+1} \arrow[r] & F_\nu \arrow[r] & F_\nu/F_{\nu+1} \arrow[r, "\epsilon"] & F_{\nu+1}[1]
\end{tikzcd}$$

A morphism of complexes δ^ν that lifts $\epsilon \circ \chi^\nu$ through $\varphi^{\nu+1}[1]$ as indicated then exists in $\mathcal{D}(\mathsf{A})$, as $P^\nu[-\nu]$ is in K^- and $\varphi^{\nu+1}[1]$ is a quasi-isomorphism. Completing the upper row by $\mathsf{cone}(\delta^\nu)$, the mapping cone over δ^ν, to an exact triangle, there exists next in the triangulated category $\mathcal{D}(\mathsf{A})$ a morphism $\varphi^\nu\colon\ \mathsf{cone}(\delta^\nu) \longrightarrow F^\nu$ as indicated so that the triple $(\varphi^{\nu+1}, \varphi^\nu, \chi^\nu)$ constitutes a morphism of exact triangles. As the other two components are isomorphisms in $\mathcal{D}(\mathsf{A})$, the same necessarily holds true for φ^ν, and, finally, that isomorphism in $\mathcal{D}(\mathsf{A})$ can be represented by an actual quasi-isomorphism of complexes, as $\mathsf{cone}(\delta^\nu)$ is by construction a complex in K^-.

It remains to observe that the morphism of complexes δ^ν involves at most a single non-zero component, represented by a morphism from $P^\nu \longrightarrow P^{\nu+1}$, due to the support of the complexes involved. Indeed, this component is nothing but the morphism induced in cohomology by the composition

$$F_\nu/F_{\nu+1} \xrightarrow{\epsilon} F_{\nu+1}[1] \twoheadrightarrow F_{\nu+1}/F_{\nu+2}[1]\,.$$

It follows in particular that $\mathbb{P}^\nu = \mathsf{cone}(\delta^\nu)$ has the desired form, with ∂^ν that single non-zero component of δ^ν, up to the sign dictated by the convention on differentials in mapping cones. This completes the inductive step.

As an aside, the reader may note that the preceding argument can as well be made directly on the level of morphisms of complexes by invoking the appropriate version of the *horseshoe lemma* to construct the quasi-isomorphism φ^ν with source $\mathbb{P}^\nu$ of the form claimed.

So far, we have constructed a diagram of morphisms of complexes

$$\begin{tikzcd} \mathbb{P}^{a+1} \arrow[r, hook] \arrow[d, "\varphi^{a+1}"', "\simeq"] & \mathbb{P}^{a} \arrow[r, hook] \arrow[d, "\varphi^{a}"', "\simeq"] & \cdots \arrow[r, hook] & \mathbb{P}^{n} \arrow[r, hook] \arrow[d, "\varphi^{n}"', "\simeq"] & \cdots \\ F^{a+1} \arrow[r, hook] & F^{a} \arrow[r, hook] & \cdots \arrow[r, hook] & F^{n} \arrow[r, hook] & \cdots \end{tikzcd}$$

and it remains to take the (essentially constant) direct limit

$$\varphi = \varinjlim_n \varphi^n \colon P = \varinjlim_n \mathbb{P}^n \xrightarrow{\simeq} \varinjlim_n F^n \cong C$$

to finish the proof. □

We add a few remarks about the essence of this degeneracy criterion.

Remark 4.5. The point is that, under the assumptions made, each differential

$$d_r^{i,n-i} \colon E_r^{i,n-i} \longrightarrow E_r^{i+r,n+1-i-r}$$

on the later pages $E_r^{\bullet,\bullet}$ for $r \geqslant 1$, by definition a morphism from a *subquotient* of $E_1^{i,n-i}$ to one of $E_1^{i+r,n+1-i-r}$, is already induced by the relevant component of $\partial^n \colon P^n = \bigoplus_i E^{i,n-i} \longrightarrow \bigoplus_i E^{i,n+1-i} = P^{n+1}$. Conversely, if there exist such morphisms ∂^n that induce the higher differentials in the spectral sequence and that satisfy $\partial^{n+1}\partial^n = 0$, then projectivity of the P^n ensures that the resulting complex is quasi-isomorphic to C, thus, constitutes a projective resolution.

Moreover, the proof shows that the construction of the projective resolution of C is effective and natural. It suffices to replace successively the connecting morphisms $F_\nu/F_{\nu+1} \longrightarrow F_{\nu+1}/F_{\nu+2}[1]$ by the morphisms $P^\nu[-\nu] \longrightarrow P^{\nu+1}[-\nu-1]$ they induce in cohomology.

Remark 4.6. It seems worthwhile to single out the simplest case. Assume that the bicomplex C not only satisfies the hypotheses of Proposition 4.4 but that furthermore there exists for each n at most one i'_n with $i_{n-1} < i'_n \leqslant i_n$ and $E_1^{i'_n,n-i'_n} \neq 0$. The spectral sequence then degenerates into a single complex

$$\cdots \longrightarrow E_1^{i'_n,n-i'_n} \xrightarrow{\partial^n} E_1^{i'_{n+1},n+1-i'_{n+1}} \longrightarrow \cdots \longrightarrow E_1^{i'_a,a-i'_a} \longrightarrow 0 \tag{4.6.1}$$

with projective terms that is quasi-isomorphic to C, thus constitutes a projective resolution of C as postulated in Proposition 4.4.

The differential ∂^n is simply obtained from the differential $d_r^{i'_n,n-i'_n} \colon E_r^{i'_n,n-i'_n} \longrightarrow E_r^{i'_{n+1},n+1-i'_{n+1}}$ on the r^{th} page of the spectral sequence, for $r = i'_{n+1} - i'_n$, through the composition

$$\partial^n \colon E_1^{i'_n,n-i'_n} \twoheadrightarrow E_r^{i'_n,n-i'_n} \xrightarrow{d_r^{i'_n,n-i'_n}} E_r^{i'_{n+1},n+1-i'_{n+1}} \hookrightarrow E_1^{i'_{n+1},n+1-i'_{n+1}}$$

where the first morphism is necessarily an epimorphism and the last one a monomorphism as the assumptions guarantee that there are no nonzero differentials with source equal to $E_{r'}^{i'_n,n-i'_n}$ or with target equal to $E_{r'}^{i'_{n+1},n+1-i'_{n+1}}$ on any earlier page $r' < r$.

5. Direct Images on the Determinantal Variety

We now come to one of the central results.

5.1. *The generic morphism.* In addition to the projective K-module F of constant rank m, let G be a second projective K-module, of constant rank $n \geqslant m$. The K-module $H = \mathrm{Hom}_K(G, F)$ is then still projective, of constant rank mn. We view H as the affine K-variety of K-rational points of $S = \mathrm{Sym}_K(H^\vee)$, locally isomorphic to a polynomial ring over K in mn variables and naturally graded by the symmetric powers which are in turn finite projective K-modules.

The projective K-modules F and G extend under $-\otimes S$ to projective S-modules $\mathcal{F}$ and $\mathcal{G}$ respectively[4]. The evaluation homomorphism $\mathrm{Hom}_K(G, F) \otimes G \longrightarrow F$ yields by adjunction the K-linear inclusion $G \hookrightarrow F \otimes H^\vee \subseteq F \otimes S$ that induces the *generic morphism* $\varphi\colon \mathcal{G} \longrightarrow \mathcal{F}$ between these projective S-modules. Taking the m^{th} exterior power over S and using that $|\mathcal{F}| = \bigwedge_S^m \mathcal{F}$ is invertible with inverse $|\mathcal{F}^\vee| = \bigwedge_S^m \mathcal{F}^\vee = |F^\vee| \otimes S$, there results an S-linear form

$$\bigwedge\nolimits_S^m \mathcal{G} \otimes_S |\mathcal{F}^\vee| \xrightarrow{(\bigwedge_S^m \varphi)\otimes_S 1} |\mathcal{F}| \otimes_S |\mathcal{F}^\vee| \longrightarrow S$$

whose image is the defining ideal of the locus where the generic morphism drops rank and whose cokernel we denote by R. Locally, $\mathrm{Spec}\, R$ is described by the vanishing of the maximal minors of the generic $(m \times n)$-matrix. The K-algebra R inherits the grading from S, and its graded components are still finite projective K-modules, as follows from the classical Gaeta-Eagon-Northcott complex [11, 9] that resolves R projectively as an S-module. In particular, R is a perfect S-module of grade equal to $n - m + 1$. The singular locus of $\mathrm{Spec}\, R$ is locally defined by the *submaximal* minors $I_{m-1}(X)$, whence has codimension $n - m + 3$ in $\mathrm{Spec}\, R$. In particular, R is smooth in codimension 2, a fact we shall exploit below.

Recall as well that $\pi\colon \mathbb{P} \longrightarrow \mathrm{Spec}\, K$ denotes the structure morphism from the projective space $\mathbb{P} = \mathbb{P}(F^\vee) \cong \mathbb{P}^{m-1}$ of K-linear forms on F to the base scheme.

Set $\mathcal{Y} = \mathbb{P} \times_{\mathrm{Spec}\, K} H$, with the canonical projections $p\colon \mathcal{Y} \longrightarrow \mathbb{P}$ and $q\colon \mathcal{Y} \longrightarrow H$. Note that q can be identified with $\pi \times_{\mathrm{Spec}\, K} H$, whence we may view it as the structure map of the projective bundle $\mathcal{Y} \cong \mathrm{Proj}_H(\mathcal{F}^\vee) \longrightarrow H$. In particular, the results of the Section 3 apply, if one replaces there K by H and F by $\mathcal{F}$.

5.2. *The incidence variety and its resolution.* Define as in the Introduction the *incidence variety*

$$\mathcal{Z} = \{\, ([\lambda], \theta) \in \mathbb{P} \times_{\mathrm{Spec}\, K} H \mid \lambda\theta = 0 \,\} \subseteq \mathcal{Y}$$

and denote by j the natural inclusion $\mathcal{Z} \longrightarrow \mathcal{Y}$. The composition $q' = qj\colon \mathcal{Z} \longrightarrow H$ is then a birational isomorphism from $\mathcal{Z}$ onto its image $q'(\mathcal{Z}) = \mathrm{Spec}\, R$, while $p' = pj\colon \mathcal{Z} \longrightarrow \mathbb{P}$ is a vector bundle (with zero section $\theta = 0$). In particular, p' is smooth, thus flat.

The vector bundle $\mathcal{Z}$ admits a compact description in terms of the bundle of differential forms $\mathcal{U} = \Omega^1_{\mathbb{P}}(1)$. Since an element of the fiber $\Omega^1(1)_\lambda$ over a closed point $\lambda \in \mathbb{P}$ sits in an exact sequence

$$0 \longrightarrow \Omega^1(1)_\lambda \longrightarrow F \longrightarrow K \longrightarrow 0\,,$$

[4] If one wishes to keep track of the S-grading, $\mathcal{G}$ should be identified with the graded S-module $G \otimes S(-1)$ generated in degree 1, while $\mathcal{F} = F \otimes S$ is generated in degree 0.

we obtain a closed point of $\mathcal{Z}$ by tensoring with $G^\vee$:

$$0 \longrightarrow \Omega^1(1)_\lambda \otimes G^\vee \longrightarrow F \otimes G^\vee \longrightarrow G^\vee \longrightarrow 0$$

and see thereby that

$$\mathcal{Z} \cong \underline{\mathrm{Spec}}\left(\mathrm{Sym}_{\mathbb{P}(F^\vee)}(\Omega^1(1)^\vee \otimes G)\right) . \tag{5.2.1}$$

The morphism $j\colon \mathcal{Z} \longrightarrow \mathcal{Y}$ is a *regular immersion* of codimension n, zero-locus of the cosection

$$\Phi\colon q^*\mathcal{G} \longrightarrow p^*(\mathcal{O}_{\mathbb{P}}(1)) = \mathcal{O}_{\mathcal{Y}}(1) \tag{5.2.2}$$

which corresponds by adjunction to the generic morphism $\mathcal{G} \longrightarrow q_*\mathcal{O}_{\mathcal{Y}}(1) \cong \mathcal{F}$ and is determined locally through

$$\Phi(q^*g_j) = \sum_{i=1}^{m} f_i \otimes x_{ij} .$$

Put differently, the S-module of sections of $\mathcal{O}_{\mathbb{P}}(1) \otimes \mathcal{O}_H$, isomorphic to $\mathcal{F}$, contains the K-linear subspace $F \otimes H^\vee$ and this subspace in turn contains G canonically. Then $\mathcal{Z}$ is the complete intersection in $\mathcal{Y} = \mathbb{P} \times_{\mathrm{Spec}\, K} H$ given locally by a basis of n sections of $G \subseteq \Gamma(\mathcal{Y}, \mathcal{O}_{\mathbb{P}}(1) \otimes \mathcal{O}_H)$.

Accordingly, the direct image $j_*\mathcal{O}_{\mathcal{Z}}$ is resolved by locally free $\mathcal{O}_{\mathcal{Y}}$-modules through the Koszul complex

$$j_*\mathcal{O}_{\mathcal{Z}} \simeq \left(\bigwedge\nolimits_{\mathcal{Y}}(q^*\mathcal{G} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathcal{O}_{\mathbb{P}}(-1)), \partial_{\Phi(-1)}\right) . \tag{5.2.3}$$

on the $\mathcal{O}_{\mathcal{Y}}$-linear form $\Phi(-1)$. As j is a finite morphism, indeed a closed immersion, $j_*\mathcal{O}_{\mathcal{Z}}$ represents already the total direct image $\mathbf{R}j_*\mathcal{O}_{\mathcal{Z}}$.

We now analyse the higher direct images $(\mathbf{R}^\nu q'_*)p'^*(\mathcal{M}_a^b(-c))$, using the degeneracy criterion from the foregoing section. As before, in view of Lemma 3.8, it suffices to treat the case $a + b \geqslant m + 1$.

Theorem 5.3. *With $\mathcal{M}_a^b(-c) = \mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(\Omega^{b-1}(b), \Omega^{a-1}(a))(-c)$ as before, the complex $(\mathbf{R}^\bullet q'_*)p'^*(\mathcal{M}_a^b(-c))$ admits a projective resolution in $\mathcal{D}(S)$ by a perfect complex that is supported on $[-n, m-1] \subseteq \mathbb{Z}$ and of amplitude at most n.*

The higher direct images $(\mathbf{R}^\nu q'_*)p'^*(\mathcal{M}_a^b(-c))$ with $\nu \neq 0$ vanish as soon as

$$c \leqslant 0 \quad \textit{or} \quad c = 1 \textit{ and } b = m \textit{ or } a = 1 \quad \textit{or} \quad c = 2, b = m \textit{ and } a = 1.$$

In these cases, the direct image $q'_*p'^*(\mathcal{M}_a^b(-c))$ admits a resolution

$$0 \longrightarrow P^{-d} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow q'_*p'^*(\mathcal{M}_a^b(-c)) \longrightarrow 0$$

by finite projective S-modules P^μ.

For $a + b \geqslant m + 1$, the non-vanishing projective modules P^μ are of the form

(5.3.1)

μ	P^μ	c
$[m-n-1,c-2]$	$\mathbf{R}^{m-1}\pi_*\mathcal{M}^b_{a(-c+\mu-m+1)}\otimes_S\bigwedge^{m-1-\mu}\mathcal{G}$	$\geqslant m-n+1$
$c-1$	$\bigoplus_{k=0}^{\min\{m-a,m-b\}}\bigwedge^{b+k}\mathcal{F}^\vee\otimes_S\bigwedge^{a-c+k}\mathcal{G}$	$\geqslant a-n$
c	$\bigoplus_{k=\max\{a,b\}}^{m}\bigwedge^{k-a}\mathcal{F}^\vee\otimes_S\bigwedge^{k-b-c}\mathcal{G}$	$[\max\{a-b-n,-n\},0]$
$[c+1,0]$	$\pi_*\mathcal{M}^b_{a(-c+\mu)}\otimes_S\bigwedge^{-\mu}\mathcal{G}$	$[-n,0]$
$[-n,0]$	$\pi_*\mathcal{M}^b_{a(-c+\mu)}\otimes_S\bigwedge^{-\mu}\mathcal{G}$	$<-n$

Accordingly, the projective dimension d of $q'_*p'^*(\mathcal{M}^b_a(-c))$, with $a+b\geqslant m+1$, is given by

d	c	(a,b)
$n-m+1$	2	$(1,m)$
$n-m+1$	1	$b=m$
$n-m+1$	$[m-n,0]$	
$-c+1$	$[a-n,m-n]$	
$-c$	$[a-b-n,a-n-1]$	$a>b$
$-c$	$[-n,a-n-1]$	$a\leqslant b$
$-c-1$	$[-n,a-b-n-1]$	$a>b$
n	$<-n$	

In particular, for arbitrary integers a,b,c, the S-module $q'_*p'^*(\mathcal{M}^b_a(-c))$ is perfect *of grade equal to $n-m+1$ for*

$$\begin{aligned}
&c=m-n-1 \text{ and } a=m \text{ or } b=1, \quad \text{or}\\
&m-n\leqslant c\leqslant 0, \quad \text{or}\\
&c=1 \text{ and } b=m \text{ or } a=1, \quad \text{or}\\
&c=2 \text{ and } b=m \text{ and } a=1.
\end{aligned}$$

Proof. Observe that $q'_*p'^*=q_*j_*j^*p^*$, whence we can calculate the desired derived direct image as

$$(\mathbf{R}q'_*)p'^*(\mathcal{M}^b_a(-c))\simeq\mathbf{R}q_*(\mathbf{R}j_*(j^*p^*\mathcal{M}^b_a(-c)))\,.$$

To evaluate the term on the right, we have first

$$j^*p^*\mathcal{M}^b_a(-c)\cong p^*\mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}\left(\Omega^{b-1}(b),\Omega^{a-1}(a)\right)(-c)\otimes_{\mathcal{O}_{\mathcal{Y}}}\mathcal{O}_{\mathcal{Z}}$$

and then

$$\mathbf{R}j_*(j^*p^*\mathcal{M}^b_a(-c))\cong p^*\mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}\left(\Omega^{b-1}(b),\Omega^{a-1}(a)\right)(-c)\otimes_{\mathcal{O}_{\mathcal{Y}}}\mathbf{R}j_*\mathcal{O}_{\mathcal{Z}}$$

by the projection formula, as $p^*\mathcal{M}^b_a(-c)$ is locally free on $\mathcal{Y}$. Replacing $\mathbf{R}j_*\mathcal{O}_{\mathcal{Z}}$ by its locally free $\mathcal{O}_{\mathcal{Y}}$-resolution described in 5.2 above, we find that $\mathbf{R}j_*(j^*p^*\mathcal{M}^b_a(-c))$ is represented in the derived category of $\mathcal{Y}$ by a (chain) complex C with terms

$$C^{-i}=p^*\mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}\left(\Omega^{b-1}(b),\Omega^{a-1}(a)\right)(-c-i)\otimes_{\mathcal{O}_{\mathcal{Y}}}p^*\pi^*\textstyle\bigwedge^iG\quad,\quad i=0,\dots,n;$$

concentrated on the interval $[-n,0]$. We can now determine the higher direct images under q_* of $\mathbf{R}j_*p'^*\mathcal{M}_a^b(-c)$ by means of the hypercohomology spectral sequence defined by this complex. The first page $E_1^{i,j}$ of that spectral sequence is concentrated in the second quadrant, supported on the rectangle $[-n,0]\times[0,m-1]$ in the (i,j)-plane, with

$$E_1^{i,j}=\mathbf{R}^jq_*(C^i)\Longrightarrow\mathbf{R}^{i+j}q'_*(p'^*\mathcal{M}_a^b(-c))\,.$$

Using the projection formula once more and noting that taking (higher) direct images commutes with flat base change, we obtain next that

$$E_1^{i,j}=\mathbf{R}^jq_*(C^i)\cong\mathbf{R}^j\pi_*(\mathcal{M}_a^b(i-c))\otimes_S\bigwedge\nolimits^{-i}\mathcal{G}\,.$$

In view of Theorem 3.9, for fixed $i\in[-n,0]$, there is at most one index j for which $E_1^{i,j}$ is not zero, and these terms are finite projective S-modules. In particular, the assumptions of Proposition 4.4 are satisfied and the hypercohomology spectral sequence degenerates into a projective resolution of $(\mathbf{R}q'_*)p'^*(\mathcal{M}_a^b(-c))$.

The first page of the spectral sequence is concentrated in total degrees $[-n,m-1]$, with at most n degrees supporting non-zero terms, whence the claims about support and amplitude of the projective resolution follow.

For the detailed analysis of the projective resolutions we exhibit their terms by means of Theorem 3.9. Recall that we assume as there that $a+b\geqslant m+1$. We proceed by cases.

(1) For (total) degree $\mu\leqslant c-2$, Theorem 3.9, or Corollary 3.24, shows that in the direct sum

$$P^\mu=\bigoplus_{j=0}^{m-1}E_1^{\mu-j,j}\cong\bigoplus_{j=0}^{m-1}\mathbf{R}^j\pi_*(\mathcal{M}_a^b(\mu-j-c))\otimes_S\bigwedge\nolimits^{j-\mu}\mathcal{G}\,,$$

only the highest occurring direct image $\mathbf{R}^{m-1}\pi_*(\mathcal{M}_a^b(\mu-m+1-c))$ can possibly be non-zero,

$$P^\mu=E_1^{\mu-m+1,m-1}\cong\mathbf{R}^{m-1}\pi_*(\mathcal{M}_a^b(-c+\mu-m+1))\otimes_S\bigwedge\nolimits^{m-1-\mu}\mathcal{G}\,.$$

Moreover, the first factor in the tensor product is indeed non-zero if, and only if, $c-\mu-1>0$ and the other factor is clearly non-zero if, and only if, $0\leqslant m-1-\mu\leqslant n$. This yields $P^\mu\neq0$ exactly for

$$m-n-1\leqslant\mu\leqslant\min\{m-1,c-2\}\,.$$

(2) In total degree $i+j=c-1$, Theorem 3.9(4) yields

$$\begin{aligned}P^{c-1}=\bigoplus_{j=0}^{m-1}E_1^{i,j}&\cong\bigoplus_{j=0}^{m-1}\mathbf{R}^j\pi_*(\mathcal{M}_a^b(-j-1))\otimes_S\bigwedge\nolimits^{j+1-c}\mathcal{G}\\&\cong\bigoplus_{j+1=a}^{\min\{a-b+m,m\}}\mathcal{F}^\vee_{b-a+j+1}\otimes_S\bigwedge\nolimits^{j+1-c}\mathcal{G}\\&=\bigoplus_{k=0}^{\min\{m-b,m-a\}}\mathcal{F}^\vee_{b+k}\otimes_S\bigwedge\nolimits^{a+k-c}\mathcal{G}\end{aligned}$$

and the second factor in the tensor product is non-zero if, and only if, $0\leqslant a+k-c\leqslant n$. Combined with the range $0\leqslant k\leqslant\min\{m-b,m-a\}$ of the summation, P^{c-1} is thus seen to be nonzero if, and only if,

$$\max\{c-a,0\} \leqslant \min\{m-a, m-b, n-a+c\}\,,$$

equivalently,

$$\max\{c,a\} \leqslant \min\{m, m+a-b, n+c\}\,.$$

In case $c \leqslant a$, this condition just becomes $a-n \leqslant c$ as $m-a, m-b \geqslant 0$.

(3) In total degree $i+j=c$ we obtain from Theorem 3.9(3) that

$$\begin{aligned} P^c = \bigoplus_{j=0}^{m-1} E_1^{i,j} &\cong \bigoplus_{j=0}^{m-1} \mathbf{R}^j\pi_*(\mathcal{M}_a^b(-j)) \otimes_S \bigwedge^{j-c}\mathcal{G} \\ &\cong \bigoplus_{j=\max\{0,a-b\}}^{m-b} \mathcal{F}^\vee_{b-a+j} \otimes_S \bigwedge^{j-c}\mathcal{G} \\ &= \bigoplus_{k=\max\{a,b\}}^{m} \mathcal{F}^\vee_{k-a} \otimes_S \bigwedge^{k-b-c}\mathcal{G}\,. \end{aligned}$$

Taking into account that the second factor in the tensor product is nonzero if, and only if, $0 \leqslant k-b-c \leqslant n$ and comparing this with the range of the summation, it follows that $P^c \neq 0$ if, and only if,

$$\max\{a, b, b+c\} \leqslant \min\{m, n+b+c\}\,.$$

If $c \leqslant 0$, this inequality becomes equivalent to $\max\{a-b-n, -n\} \leqslant c \leqslant 0$. Note also that $P^c = 0$ for $b+c > m$.

(4) Finally assume that the total degree satisfies $\mu = i+j > c$. In that case thus $c-i < j$ and Corollary 3.24 shows $E_1^{i,j} = 0$ for $j \neq 0$, whence

$$P^\mu = E^{\mu,0} \cong \pi_*(\mathcal{M}_a^b(\mu - c)) \otimes_S \bigwedge^{-\mu}\mathcal{G}$$

In turn, this term is non-zero if, and only if, $\max\{-n, c+1\} \leqslant \mu \leqslant 0$.

It remains to exhibit when $P^\mu \neq 0$ for some $\mu > 0$. By case (1), this will occur if $0 < \min\{m-1, c-2\}$, thus, for $c > 2$ (and $m \geqslant 2$).

If $c = 2$, then $P^1 = P^{c-1} \neq 0$ if, and only if, $\max\{2,a\} \leqslant \min\{m, m-b+a, n+2\}$ by case (2) above. As we always have $\max\{2,a\} \leqslant m < n+2$ and $a \leqslant m-b+a$, the inequality fails only for $a = 1, b = m$. In the latter case, indeed each $P^\mu = 0$ for $\mu > 0$.

If $c = 1$, then the above results yield immediately $P^\mu = 0$ for all $\mu > 1$, and case (3) shows that $P^1 = 0$ if, and only if, $\max\{a, b, b+1\} = \max\{a, b+1\} > \min\{m, n+b+1\} = m$, which in turn holds if, and only if, $b = m$. □

Remark 5.4. For any $n \geqslant m$, a projective resolution for $q'_*p'^*\mathcal{M}_a^b$ cannot be shorter than displayed. Inspecting P^{-n+m-1}, this implies, in a backhanded way, that $\mathbf{R}^{m-1}\pi_*\mathcal{M}_a^b(-c) \neq 0$ for each $c > m$, whence also $\pi_*\mathcal{M}_a^b(-c) \neq 0$ for $c < 0$.

Example 5.5. To derive an S-presentation for $q'_*p'^*\Omega^{a-1}(a)$, consider that we have

$$\begin{aligned} \mathcal{M}_a^m &= \mathscr{H}om_{\mathcal{O}_\mathbb{P}}\left(\Omega^{m-1}(m), \Omega^{a-1}(a)\right) \\ &= \mathscr{H}om_{\mathcal{O}_\mathbb{P}}\left(\bigwedge^m F \otimes \mathcal{O}_\mathbb{P}, \Omega^{a-1}(a)\right) \\ &= \bigwedge^m F^\vee \otimes \Omega^{a-1}(a) \end{aligned}$$

and hence

$$\Omega^{a-1}(a) = |F| \otimes \mathcal{M}_a^m\,, \tag{5.5.1}$$

where $|F|$ is the determinant of F; recall 3.5. From the second and the third line of the table (5.3.1), applied with $c = 0$, we find that $q'_*p'^*\mathcal{M}_a^m$ has a presentation

$$\bigwedge^m \mathcal{F}^\vee \otimes_S \bigwedge^a \mathcal{G} \longrightarrow \bigwedge^{m-a} \mathcal{F}^\vee \longrightarrow q'_*p'^*\mathcal{M}_a^m \longrightarrow 0\,.$$

Tensoring with $|F|$ we get a presentation

$$\bigwedge^a \mathcal{G} \xrightarrow{\rho} \bigwedge^a \mathcal{F} \longrightarrow q'_*p'^*\Omega^{a-1}(a) \longrightarrow 0\,.$$

We confirm the identity of ρ below in Theorem 6.2.

6. From Algebra to Geometry

We now use the homological results from Sections 3–5 to prove the results asserted in the Introduction.

6.1. *The non-commutative desingularization.* We retain the notations from 5.1 and 5.2, but from now on K will always be a *field.* As in the Introduction, we put

$$M_a = \operatorname{cok} \textstyle\bigwedge_S^a \varphi$$

for $1 \leqslant a \leqslant m$, and $M = \bigoplus_a M_a$. Set $E = \operatorname{End}_R(M)$, our intended non-commutative desingularization of $\operatorname{Spec} R$.

First we obtain a geometric description of M_a.

Theorem 6.2. *There is an isomorphism $q'_*(p'^*\Omega^{a-1}(a)) \cong M_a$, which fits in the following commutative diagram*

$$\begin{tikzcd}
q'_*q'^*\bigwedge^a \mathcal{F} \arrow[r, two heads] & q'_*p'^*\Omega^{a-1}(a) \\
\bigwedge^a \mathcal{F} \arrow[u, two heads] \arrow[r, two heads] & M_a \arrow[u, "\cong"']
\end{tikzcd}$$

where the leftmost vertical map is the canonical one, the lower horizontal map comes from the definition of M_a, and the upper horizontal map is derived from the exact sequence in 3.5.1.

Proof. Let $i\colon \operatorname{Spec} R \longrightarrow \operatorname{Spec} S$ be the inclusion. We will construct a more elaborate version of the claimed diagram

$$\begin{tikzcd}
i^*q'_*\bigwedge^a q'^*(\mathcal{G}) \arrow[r, "i^*(q_*\bigwedge^a q'^*(\varphi))"] & i^*q'_*\bigwedge^a q'^*(\mathcal{F}) \arrow[r] & i^*q'_*(p'^*\Omega^{a-1}(a)) \arrow[r] & 0 \\
i^*\bigwedge^a \mathcal{G} \arrow[u, "\cong"] \arrow[r, "i^*(\bigwedge^a \varphi)"'] & i^*\bigwedge^a \mathcal{F} \arrow[u, "\cong"'] \arrow[r] & i^*M_a \arrow[u, "\cong"'] \arrow[r] & 0
\end{tikzcd} \tag{6.2.1}$$

where the two leftmost vertical maps are the canonical ones.

For brevity we will drop below most of the applications of i^* from the notations.

Let $H_0 \subset H$ be the locus where the rank of φ is exactly $m-1$ and put $\mathcal{Z}_0 = (q')^{-1}(H_0)$. Then q' restricted to $\mathcal{Z}_0$ is an isomorphism.

The map $\varphi\colon \mathcal{F} \longrightarrow \mathcal{G}$ pulls back to a map $q'^*(\varphi)\colon q'^*(\mathcal{G}) \longrightarrow q'^*(\mathcal{F})$. By looking at fibers it is easy to see that it factors as

$$q'^*(\varphi)\colon \quad q'^*(\mathcal{G}) \longrightarrow p'^*(\Omega(1)) \hookrightarrow p'^*\pi^*F = q'^*(\mathcal{F})\,. \tag{6.2.2}$$

Since the exterior product preserves subbundles we get an factorization

$$\textstyle\bigwedge^a q'^*(\varphi)\colon \quad \bigwedge^a q'^*(\mathcal{G}) \longrightarrow p'^*(\Omega^a(a)) \hookrightarrow \bigwedge^a p'^*\pi^*F = \bigwedge^a q'^*(\mathcal{F})$$

and hence combining this with the pullback of a suitably shifted version of (3.5.1) under p' we get a complex

$$\textstyle\bigwedge^a q'^*(\mathcal{G}) \xrightarrow{\bigwedge^a q'^*(\varphi)} \bigwedge^a q'^*(\mathcal{F}) \longrightarrow p'^*(\Omega^{a-1}(a)) \longrightarrow 0\,. \tag{6.2.3}$$

Since the first map in (6.2.2) is an epimorphism when restricted to $\mathcal{Z}_0$ and since exterior powers also preserve epimorphisms, we get that (6.2.3) is exact when restricted to $\mathcal{Z}_0$.

It follows that we have a complex

$$\textstyle q'_*\bigwedge^a q'^*(\mathcal{G}) \xrightarrow{q'_*\bigwedge^a q'^*(\varphi)} q'_*\bigwedge^a q'^*(\mathcal{F}) \longrightarrow q'_*p'^*\Omega^{a-1}(a) \longrightarrow 0 \tag{6.2.4}$$

exact on H_0. Comparing this with the right-exact sequence on $\operatorname{Spec} R$

$$\textstyle i^*\bigwedge^a \mathcal{G} \xrightarrow{i^*\bigwedge^a \varphi} i^*\bigwedge^a \mathcal{F} \longrightarrow M_a \longrightarrow 0$$

we obtain (6.2.1), with a uniquely defined rightmost vertical map. It remains to show that the vertical maps are isomorphisms. We will only consider the rightmost one, as the others are similar but easier.

Since (6.2.4) is exact on H_0 we find

$$q'_*p'^*\Omega^{a-1}(a)\big|_{H_0} = M_a\big|_{H_0}\,.$$

Now $q'_*p'^*\Omega^{a-1}(a)$ is R-torsion free and M_a is maximal Cohen-Macaulay over R (and hence R-reflexive) by [7, Corollary 2.6]. Since the codimension of the complement of H_0 in $\operatorname{Spec} R$ is at least 2 we obtain that the induced map $M_a \longrightarrow q'_*p'^*\Omega^{a-1}(a)$ is an isomorphism. $\square$

6.3. *A tilting bundle.* Put $\mathcal{T}_a = p'^*\Omega^{a-1}(a)$ and $\mathcal{T} = \bigoplus_{a=1}^m \mathcal{T}_a$, bundles on the incidence variety $\mathcal{Z}$. It follows from Theorem 6.2 that $\operatorname{End}_R(q'_*\mathcal{T}) \cong \operatorname{End}_R(M) = E$. We can now prove Theorems A and C from the Introduction.

Theorem 6.4. *We have $\mathcal{T}^\perp = 0$ in $\mathcal{D}(\operatorname{Qch}(\mathcal{Z}))$ and $\operatorname{Ext}^i_{\mathcal{O}_\mathcal{Z}}(\mathcal{T},\mathcal{T}) = 0$ for $i > 0$. In other words, $\mathcal{T}$ is a classical tilting bundle on $\mathcal{Z}$ in the sense of* [13].

Proof. The condition $\mathcal{T}^\perp = 0$ follows immediately by considering the adjoint pair (p'^*, p'_*) and the fact, due to Beĭlinson [1], that $\bigoplus_{a=1}^m \Omega^{a-1}(a)$ is a tilting bundle on $\mathbb{P}(F^\vee)$. The vanishing of Ext follows from Theorem 5.3 applied with $c = 0$. $\square$

Theorem 6.5. *We have $E \cong \operatorname{End}_{\mathcal{O}_\mathcal{Z}}(\mathcal{T})$. Furthermore E is noetherian on both sides, is finite over its centre, has finite global dimension and is a maximal Cohen-Macaulay R-module.*

Proof. Put $E' = \operatorname{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T})$. Since $\mathcal{T}$ is a tilting bundle on $\mathcal{Z}$ we obtain $\mathcal{D}^b(\operatorname{coh}(\mathcal{Z})) \cong \mathcal{D}^b_f(E')$. Since $\mathcal{Z}$ is smooth it follows from [13, Theorem 7.6] that E' has finite global dimension.

From Theorem 5.3, applied again with $c = 0$, it follows that E' is Cohen-Macaulay.

We now have maps

$$E' = \operatorname{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T}) \longrightarrow \operatorname{End}_S(q'_*\mathcal{T}) \cong \operatorname{End}_S(M) = E\,. \tag{6.5.1}$$

The locus where φ is not an isomorphism has codimension at least 2 in both $\operatorname{Spec} R$ and $\mathcal{Z}$, whence (6.5.1) is an isomorphism in codimension one. Since both source and target of (6.5.1) are reflexive (the former e.g. by [22, Lemma 4.2.1]) we obtain that (6.5.1) is an isomorphism. $\square$

7. The Quiverized Clifford Algebra

In this section we compute the algebra structure of the non-commutative desingularization E defined in 6.1, giving in particular an explicit description of E as a path algebra of a certain quiver with relations derived in a natural way from a Clifford algebra.

7.1. *Notation.* Our setting will be as in Section 6, so in particular K is a field. In addition we fix ordered bases $\{f_1, \dots, f_m\}$ and $\{g_1, \dots, g_n\}$ for F and G, and let $\{\lambda_1, \dots, \lambda_m\}$, $\{\mu_1, \dots, \mu_n\}$ be the associated dual bases for $F^\vee$ and $G^\vee$.

We again set $S = \operatorname{Sym}_K(\operatorname{Hom}_K(G, F)^\vee) = \operatorname{Sym}_K(F^\vee \otimes G)$, which is canonically isomorphic to the polynomial ring over K in the variables $x_{ij} = \lambda_i \otimes g_j$. We let X be the generic $(m \times n)$-matrix with entries $(x_{ij})_{ij}$, so that X is the matrix of the map φ when expressed in terms of the bases $\{g_1, \dots, g_n\}$, $\{f_1, \dots, f_m\}$.

By $\operatorname{Cliff}_S(q_\varphi)$ we will denote the Clifford algebra over S associated to the quadratic form $q_\varphi \colon \mathcal{F}^\vee \oplus \mathcal{G} \longrightarrow S$ which is such that $q_\varphi(\lambda, g) = \lambda(\varphi(g))$. Concretely $\operatorname{Cliff}_S(q_\varphi)$ is the S-algebra generated by $F^\vee$ and G subject to the relations

$$\begin{aligned} \lambda_i\lambda_j + \lambda_j\lambda_i &= 0 = \lambda_i^2 && \text{for } i, j = 1, \dots, m;\\ g_ig_j + g_jg_i &= 0 = g_i^2 && \text{for } i, j = 1, \dots, n; \text{ and}\\ \lambda_ig_j + g_j\lambda_i &= x_{ij} && \text{for } i = 1, \dots, m,\ j = 1, \dots, n\,. \end{aligned}$$

7.2. *Quivers.* Let Γ be a quiver—a directed graph—on finitely many vertices $\{1, \dots, r\}$. Let D be a commutative ring (below it will be K or S). Denote by Γ_{ij} the free D-module with basis the set of paths in Γ from vertex i to vertex j, including the trivial path e_u at each vertex u. The direct sum $D\Gamma = \bigoplus_{i,j} \Gamma_{ij}$ is naturally a D-algebra, the *path algebra* of Γ, with multiplication $\Gamma_{jk} \otimes \Gamma_{ij} \longrightarrow \Gamma_{ik}$ given by concatenation of paths where possible, and all other products trivial. (Observe the indexing: we write our paths in functional order.) The paths e_u are idempotent and $\sum_u e_u$ is the identity element in $D\Gamma$, conveniently denoted by 1. Below we will also consider quivers with an infinite number of vertices (indexed from $-\infty$ to ∞). In that case $D\Gamma$ does not have a unit element, but the e_u are local units.

Let $I \subseteq D\Gamma$ be a two-sided ideal. The pair (Γ, I) is called a *quiver with relations,* and the quotient $D\Gamma/I$ its path algebra with relations. The relations I will often be understood and dropped from the notation.

7.3. *Quiverization.* If A is a $\mathbb{Z}$-graded algebra then we define the *infinite quiverization*[5] as the bigraded algebra without unit $Q_\infty(A) = \bigoplus_{i,j\in\mathbb{Z}} A_{j-i}$ with multiplication coming from the multiplication in A: $A_{k-j} \times A_{j-i} \longrightarrow A_{k-i}$. The term "quiverization" is meant to be informal, indicating that $Q_\infty(A)$ can often be advantageously represented as a path algebra of a quiver with relations on a set of vertices indexed by $\mathbb{Z}$. If M is a $\mathbb{Z}$-graded A-module then we may view M as as left $Q_\infty(A)$-module through the action $A_{j-i} \times M_i \longrightarrow M_j$. We will denote this $Q_\infty(A)$-module by $Q(M)$.

For every $i \in \mathbb{Z}$ we have $1 \in A_0 = Q_\infty(A)_{ii}$. This is an idempotent in $Q_\infty(A)$ which we denote by e_i. The *quiverization* $Q_r(A)$ *of order* r of A is defined as the quotient $Q_\infty(A)/\sum_{i\notin[1,r]} Q_\infty(A)e_iQ_\infty(A)$. It is easy to see that $Q(M)$ is a right $Q_r(A)$-module provided the grading of M is supported only in degrees $1,\dots,r$. We can often represent $Q_r(A)$ naturally by a quiver with vertices $[1,r]$.

The following lemma is trivial to prove.

Lemma 7.4. *The functor $M \rightsquigarrow Q(M)$ defines an equivalence of categories between respectively:*

(1) The category of graded A-modules and the category of graded $Q_\infty(A)$-modules N such that $N = \bigoplus_i e_iN$.

(2) The category of graded A-modules whose support is concentrated in degrees $1,\dots,r$ and the category of $Q_r(A)$-modules.

7.5. *The doubled Beĭlinson quiver.* It is clear that $\operatorname{Cliff}_S(q_\varphi)$ is bigraded by $\deg F = (1,0)$, $\deg G = (0,1)$. In this paper we consider two induced $\mathbb{Z}$-gradings. For the first one (labeled "the $\mathbb{Z}$-grading") we put $\deg F^\vee = -1$, $\deg G = 1$. For the second one ("the $\mathbb{N}$-grading") we put $\deg F^\vee = \deg G = 1$.

The *quiverized Clifford algebra* on $F^\vee$ and G is defined as $C = Q_m(\operatorname{Cliff}_S(q_\varphi))$ with $\operatorname{Cliff}_S(q_\varphi)$ considered as being graded by the $\mathbb{Z}$-grading. Note that C is still naturally bigraded.

The S-algebra C can be represented as the path algebra with relations over S of the *doubled Beĭlinson quiver*:

$$\widetilde{\mathsf{Q}}: \quad \begin{tikzcd}[column sep=huge] 1 \arrow[r, bend right=15, "g_1"'] \arrow[r, bend right=50, "g_n"'] & 2 \arrow[l, bend right=15, "\lambda_1"'] \arrow[l, bend right=50, "\lambda_m"'] \arrow[r, bend right=15, "g_1"'] \arrow[r, bend right=50, "g_n"'] & \cdots\cdots \arrow[l, bend right=15, "\lambda_1"'] \arrow[l, bend right=50, "\lambda_m"'] \arrow[r, bend right=15, "g_1"'] \arrow[r, bend right=50, "g_n"'] & m \arrow[l, bend right=15, "\lambda_1"'] \arrow[l, bend right=50, "\lambda_m"'] \end{tikzcd}$$

Note that g_i, λ_j serve as the label for $m-1$ different arrows. If there is confusion possible then we use notations like pe_u or e_vp to indicate explicitly the starting or ending point of the path p.

The a,b graded piece C_{ab} of C consists of paths from a to b, thus $C_{ab} = e_bCe_a$.

The relations (with coefficients in S) on $\widetilde{\mathsf{Q}}$ are directly derived from those of $\operatorname{Cliff}_S(q_\varphi)$:

$$\begin{aligned} \lambda_i\lambda_j + \lambda_j\lambda_i &= 0 = \lambda_i^2 && \text{for } i,j = 1,\dots,m; \\ g_ig_j + g_jg_i &= 0 = g_i^2 && \text{for } i,j = 1,\dots,n; \text{ and} \\ \lambda_ig_j + g_j\lambda_i &= x_{ij} && \text{for } i = 1,\dots,m,\ j = 1,\dots,n \end{aligned}$$

[5]The knowledgeable reader will note we are basically using the formalism of $\mathbb{Z}$-algebras here. See e.g. [5].

where we use the convention that whenever there are paths in such relations that are not defined we silently drop them. This means that the relation of the third type associated to vertex 1 is in fact $\lambda_i g_j = x_{ij}$ and the one associated to vertex m is $g_j\lambda_i = x_{ij}$.

These relations generate an ideal $\mathcal{J}$ in the path-algebra $S\widetilde{\mathsf{Q}}$ and we have $C = S\widetilde{\mathsf{Q}}/\mathcal{J}$.

For further reference we note that C has an involution

$$\lambda_i \mapsto \lambda_i, \quad g_j \mapsto g_j, \quad e_i \mapsto e_{m+1-i} \tag{7.5.1}$$

which sends C_{ab} to $C_{m+1-b,m+1-a}$.

Remark 7.6. If we prefer to do so we may work over the ground field K instead of over S. We find $C = K\widetilde{\mathsf{Q}}/\mathcal{J}'$ where $\mathcal{J}'$ is generated by the relations

$$\begin{aligned}
\lambda_i\lambda_j + \lambda_j\lambda_i = 0 = \lambda_i^2 & \qquad \text{for } i,j = 1,\dots,m;\\
g_ig_j + g_jg_i = 0 = g_i^2 & \qquad \text{for } i,j = 1,\dots,n;\\
\lambda_k(\lambda_i g_j + g_j\lambda_i) = (\lambda_i g_j + g_j\lambda_i)\lambda_k & \qquad \text{for } i,k = 1,\dots,m,\ j = 1,\dots,n; \text{ and}\\
g_l(\lambda_i g_j + g_j\lambda_i) = (\lambda_i g_j + g_j\lambda_i)g_l & \qquad \text{for } i = 1,\dots,m,\ j,l = 1,\dots,n.
\end{aligned}$$

The isomorphisms between the former presentation of C and this one are given by

$$S\widetilde{\mathsf{Q}}/\mathcal{J} \longrightarrow K\widetilde{\mathsf{Q}}/\mathcal{J}' \colon \lambda_i \mapsto \lambda_i, g_j \mapsto g_j, x_{ij} \mapsto \lambda_i g_j + g_j\lambda_i$$

and

$$K\widetilde{\mathsf{Q}}/\mathcal{J}' \longrightarrow S\widetilde{\mathsf{Q}}/\mathcal{J} \colon \lambda_i \mapsto \lambda_i, g_j \mapsto g_j\,.$$

It follows that, when considered as a K-algebra, C has cubic relations.

7.7. *A Clifford action on M.* We construct a natural map $C \longrightarrow E = \operatorname{End}_R(M)$. To describe a map $C \longrightarrow E$ we have to put a left C-module structure on M, and according to Lemma 7.4 it is sufficient to construct an action of $\operatorname{Cliff}_S(q_\varphi)$ on M.

An S-endomorphism (or equivalently R-endomorphism) of $M = \bigoplus_{a=1}^m M_a = \bigoplus_{a=1}^m \operatorname{cok}(\bigwedge^a\varphi)$ is obtained from a pair of morphisms α, β rendering the diagram

$$\begin{tikzcd}
\bigwedge\mathcal{G} \arrow[r, "\bigwedge\varphi"] \arrow[d, "\beta"'] & \bigwedge\mathcal{F} \arrow[r] \arrow[d, "\alpha"] & M \arrow[r] \arrow[d, dotted] & 0\\
\bigwedge\mathcal{G} \arrow[r, "\bigwedge\varphi"'] & \bigwedge\mathcal{F} \arrow[r] & M \arrow[r] & 0
\end{tikzcd} \tag{7.7.1}$$

commutative (putting $\bigwedge^0\mathcal{F} = \bigwedge^0\mathcal{G} = S$ and $\bigwedge^0\varphi = \mathrm{id}_S$). We construct such α, β as (super-)differential operators on $\bigwedge\mathcal{F}$ and $\bigwedge\mathcal{G}$.

(1) For $\lambda \in \mathcal{F}^\vee$, define skew-derivations $\partial_\lambda \colon \bigwedge\mathcal{F} \longrightarrow \bigwedge\mathcal{F}$ of degree -1 in $\mathcal{F}$ by (left) contraction $\lambda \curlyvee -$; explicitly, for an element $f^1 \wedge \cdots \wedge f^a \in \bigwedge^a\mathcal{F}$,

$$\partial_\lambda(f^1 \wedge \cdots \wedge f^a) = \sum_{j=1}^{a} (-1)^{j-1}\lambda(f^j)(f^1 \wedge \cdots \wedge \widehat{f^j} \wedge \cdots \wedge f^a)\,.$$

Then ∂_λ extends as well to a skew derivation $\partial_{\lambda\varphi} \colon \bigwedge\mathcal{G} \longrightarrow \bigwedge\mathcal{G}$. Putting $(\alpha, \beta) = (\partial_\lambda, \partial_{\lambda\varphi})$ makes (7.7.1) commute. Denote the induced endomorphism of M again by ∂_λ.

(2) For $g \in \mathcal{G}$, define $\theta_g \colon \bigwedge \mathcal{G} \longrightarrow \bigwedge \mathcal{G}$ by the exterior multiplication $\theta_g(-) = g \wedge -$. We have an induced map $\theta_{\varphi(g)} \colon \bigwedge \mathcal{F} \longrightarrow \bigwedge \mathcal{F}$. Putting $(\alpha, \beta) = (\theta_{\varphi(g)}, \theta_g)$ makes (7.7.1) commute. We denote the induced endomorphism of M also by θ_g.

Write $\partial_i = \partial_{\lambda_i}$, $\theta_j = \theta_{g_j}$. It is easy to see that we have

(1) $\partial_i \partial_j + \partial_j \partial_i = 0 = \partial_i^2$ and $\theta_i \theta_j + \theta_j \theta_i = 0 = \theta_i^2$; and
(2) $\partial_i \theta_j + \theta_j \partial_i = \partial_i(\varphi(g_j)) = x_{ij}$.

and hence we have defined an action of $\mathrm{Cliff}_S(q_\varphi)$ on M.

We will prove below in Theorem 7.17 that the morphism $C \longrightarrow \mathrm{End}_R(M)$ defined by the action above is an isomorphism, sending C_{ab} to $\mathrm{Hom}_R(M_b, M_a)$. Our avenue of proof once more proceeds by translating to geometry, where we define an action of the Clifford algebra C on the tilting bundle $\mathcal{T}$. We prove (Proposition 7.14) that the two actions are compatible with the isomorphism $E \cong \mathrm{End}_{\mathcal{O}_\mathcal{Z}}(\mathcal{T})$ from Theorem 6.5, and then (Theorem 7.15) that this second action gives an isomorphism $C \longrightarrow \mathrm{End}_{\mathcal{O}_\mathcal{Z}}(\mathcal{T})$.

7.8. *An S-presentation for C.* In this section we prove a partial technical result (Lemma 7.12) which we will use in the proof of Theorem 7.15.

Definition 7.9. With $\{\lambda_1, \ldots, \lambda_m\}$ and $\{g_1, \ldots, g_n\}$ the fixed bases of $F^\vee$ and G, let Q^∞ be the doubly infinite quiver over S

$$\begin{tikzcd}[column sep=huge]
\cdots \arrow[r, bend right=20, "g_1"'] \arrow[r, bend right=50, "g_n"'] & 0 \arrow[l, bend right=20, "\lambda_1"'] \arrow[l, bend right=50, "\lambda_m"'] \arrow[r, bend right=20, "g_1"'] \arrow[r, bend right=50, "g_n"'] & 1 \arrow[l, bend right=20, "\lambda_1"'] \arrow[l, bend right=50, "\lambda_m"'] \arrow[r, bend right=20, "g_1"'] \arrow[r, bend right=50, "g_n"'] & \cdots \arrow[l, bend right=20, "\lambda_1"'] \arrow[l, bend right=50, "\lambda_m"'] \arrow[r, bend right=20, "g_1"'] \arrow[r, bend right=50, "g_n"'] & m \arrow[l, bend right=20, "\lambda_1"'] \arrow[l, bend right=50, "\lambda_m"'] \arrow[r, bend right=20, "g_1"'] \arrow[r, bend right=50, "g_n"'] & \cdots \arrow[l, bend right=20, "\lambda_1"'] \arrow[l, bend right=50, "\lambda_m"']
\end{tikzcd}$$

with relations

$$\lambda_i \lambda_j + \lambda_j \lambda_i = 0 = \lambda_i^2$$
$$g_i g_j + g_j g_i = 0 = g_i^2$$
$$\lambda_i g_j + g_j \lambda_i = x_{ij}\,.$$

We define $C^\infty = Q(\mathrm{Cliff}_S(q_\varphi))$. Then C^∞ is the S-path algebra of Q^∞ with relations as above. Of course C^∞ is again naturally graded by $C^\infty_{ab} = e_b C^\infty e_a$, and surjects onto C.

Verification of the following version of a Poincaré-Birkhoff-Witt (PBW) basis for C^∞ is routine (and follows formally from the existence of a similar basis for $\mathrm{Cliff}_S(q_\varphi)$). Recall that we write paths in Q_∞ in functional order.

Lemma 7.10. *The algebra C^∞ is free as an S-module. More precisely, a basis for the graded piece C^∞_{ab} consists of paths*

$$e_b \lambda_{\beta_b} \lambda_{\beta_{b+1}} \cdots \lambda_{\beta_l} g_{\alpha_l} g_{\alpha_{l-1}} \cdots g_{\alpha_a} e_a \tag{7.10.1}$$

with $\alpha_a > \alpha_{a+1} > \cdots > \alpha_l$ and $\beta_l < \beta_{l-1} < \cdots < \beta_b$.

We will refer to writing an element of C^∞ in terms of this basis as the "PBW expansion for the ordering $\lambda_m < \cdots < \lambda_1 < g_1 < \cdots < g_n$. There is a similar PBW expansion with the roles of g_i,λ_j reversed.

Proposition 7.11. *Let D be the kernel of the surjection $C^\infty \longrightarrow C$. The graded piece D_{ab} is S-generated by two types of paths: those leaving $[1,m]$ to the right*

$$e_b\lambda_{\beta_b}\lambda_{\beta_{b+1}}\cdots\lambda_{\beta_l}g_{\alpha_l}g_{\alpha_{l-1}}\cdots g_{\alpha_a}e_a \tag{7.11.1}$$

with $l > m$, $\alpha_a > \alpha_{a+1} > \cdots > \alpha_l$, and $\beta_l < \beta_{l-1} < \cdots < \beta_b$; and those leaving $[1,m]$ to the left

$$e_b g_{\alpha_b}g_{\alpha_{b-1}}\cdots g_{\alpha_l}\lambda_{\beta_l}\lambda_{\beta_{l+1}}\cdots\lambda_{\beta_a}e_a \tag{7.11.2}$$

with $l < 1$, $\beta_a < \beta_{a-1} < \cdots < \beta_l$, and $\alpha_l > \alpha_{l+1} > \cdots > \alpha_b$.

Proof. We need to prove that the paths (7.11.1) and (7.11.2) generate D_{ab}. To this end, we claim that with the natural identifications $\bigwedge^k \mathcal{F}^\vee \subseteq C^\infty_{a,a-k}$ and $\bigwedge^k \mathcal{G} \subseteq C^\infty_{a,a+k}$ in mind,

$$C^\infty_{lb}\cdot C^\infty_{al} \subseteq \sum_{k\geqslant l}\bigwedge\nolimits^{k-b}\mathcal{F}^\vee\cdot\bigwedge\nolimits^{k-a}\mathcal{G} \tag{7.11.3}$$

and, symmetrically,

$$C^\infty_{lb}\cdot C^\infty_{al} \subseteq \sum_{k\leqslant l}\bigwedge\nolimits^{k-b}\mathcal{G}\cdot\bigwedge\nolimits^{k-a}\mathcal{F}^\vee\,.$$

Indeed, by Lemma 7.10, any element of $C^\infty_{lb}\cdot C^\infty_{al}$ is a linear combination of paths of the form $e_b\boldsymbol{\lambda}\boldsymbol{g}e_l\boldsymbol{\lambda}'\boldsymbol{g}'e_a$, where $\boldsymbol{\lambda},\boldsymbol{\lambda}'$, respectively $\boldsymbol{g},\boldsymbol{g}'$ represent products of λ_i, respectively g_j. The length of the path $\boldsymbol{\lambda}$ is not less than $l-b$, while that of $\boldsymbol{g}'$ is not less than $l-a$. Applying Lemma 7.10 to the product $\boldsymbol{g}\boldsymbol{\lambda}'$ then gives the first containment. The other follows similarly. □

The presentation in Proposition 7.11 is not minimal for the $\mathbb{N}$-grading on S. We next give a slightly smaller presentation, which is sufficient for our proof of Theorem 7.17, even though it is still not minimal. For the best result see Proposition 9.3.

Lemma 7.12. *The graded piece C_{ab} has an S-free presentation of the form*

$$Q\oplus P_1 \xrightarrow{\rho} P_0 \longrightarrow C_{ab}\longrightarrow 0\,, \tag{7.12.1}$$

where

- $P_0 = \bigoplus_{\max\{a,b\}\leqslant k\leqslant m}\bigwedge_S^{k-b}\mathcal{F}^\vee\otimes\bigwedge_S^{k-a}\mathcal{G}$
- $P_1 = \bigoplus_{0\geqslant l\geqslant \max\{a-m,b-m\}}\bigwedge_S^{b-l}\mathcal{G}\otimes\bigwedge_S^{a-l}\mathcal{F}^\vee$
- $Q = \bigoplus_{\max\{a-m,b-m\}>l\geqslant\max\{a-m,b-n\}}\bigwedge_S^{b-l}\mathcal{G}\otimes\bigwedge_S^{a-l}\mathcal{F}^\vee$

and the map ρ is the restriction of the inclusion of D into C^∞. Furthermore $\ker(\rho\,|_{P_1})\subseteq S_{>0}P_1$.

Proof. Our starting point is the free presentation of C_{ab} given in Proposition 7.11. It takes the form (remember once again that paths are written in functional order) $e_bDe_a \hookrightarrow e_bC^\infty e_a$, where

$$e_bC^\infty e_a = \bigoplus_{\max\{a,b\}\leqslant k\leqslant\min\{a+n,b+m\}}\bigwedge\nolimits_S^{k-b}\mathcal{F}^\vee\otimes\bigwedge\nolimits_S^{k-a}\mathcal{G}$$

and

$$e_b D e_a = \left(\bigoplus_{m<k\leqslant \min\{a+n,b+m\}} \bigwedge\nolimits_S^{k-b} \mathcal{F}^\vee \otimes \bigwedge\nolimits_S^{k-a} \mathcal{G} \right) \oplus \left(\bigoplus_{1>l\geqslant \max\{a-m,b-n\}} \bigwedge\nolimits_S^{b-l} \mathcal{G} \otimes \bigwedge\nolimits_S^{a-l} \mathcal{F}^\vee \right).$$

In the resulting presentation there is some cancellation, which simplifies things to

$$\bigoplus_{1>l\geqslant \max\{a-m,b-n\}} \bigwedge\nolimits_S^{b-l} \mathcal{G} \otimes \bigwedge\nolimits_S^{a-l} \mathcal{F}^\vee \longrightarrow \bigoplus_{\max\{a,b\}\leqslant k\leqslant m} \bigwedge\nolimits_S^{k-b} \mathcal{F}^\vee \otimes \bigwedge\nolimits_S^{k-a} \mathcal{G}$$

which is (7.12.1).

Now we prove the additional claim of the lemma. Assume that we have

$$\rho\left(\sum_{\alpha,l=\max\{a-m,b-m\}}^{0} s_{l,\alpha} p_{l,\alpha} \right) = 0$$

with $p_{l,\alpha} \in \bigwedge_S^{b-l} \mathcal{G} \otimes \bigwedge_S^{a-l} \mathcal{F}^\vee$ and $s_{l,\alpha} \in S$. This can be rewritten as an identity in C^∞

$$\sum_{\alpha,l=\max\{a-m,b-m\}}^{0} s_{l,\alpha} p_{l,\alpha} = \sum_{\beta,k\geqslant m+1} t_{k,\beta} q_{k,\beta} \tag{7.12.2}$$

with $q_{k,\beta} \in \bigwedge_S^{k-b} \mathcal{F}^\vee \otimes \bigwedge_S^{k-a} \mathcal{G}$ and $t_{k,\beta} \in S$. We may assume that the $s_{l,\alpha}$, $t_{k,\beta}$ are homogeneous for the $\mathbb{N}$-grading.

Choose l' maximal such that there exists $s_{l',\alpha} \neq 0$. We have to show that $s_{l',\alpha} \in S_{>0}$ for all α corresponding to this l'. Assume on the contrary that there is some α' such that $s_{l',\alpha'} \notin S_{>0}$.

By our restriction on l we have $b - l \leqslant m$, $a - l \leqslant m$ in the expression for $p_{l,\alpha}$. Right-multiplying (7.12.2) by a suitable product of the λ_j and left-multiplying by a suitable product of the g_i, we obtain an identity (using (7.11.3)) of paths starting and ending in some vertex $v \in [1, m]$

$$\sum_{1\leqslant i_1<\ldots<i_m\leqslant n} s'_{i_1\ldots i_m} g_{i_1} \cdots g_{i_m} \lambda_1 \cdots \lambda_m = \sum_{\beta'} t'_{\beta'} q'_{\beta'} \tag{7.12.3}$$

where $s'_{i_1\ldots i_m} \in S$ and at least one $s'_{i_1\ldots i_m} \notin S_{>0}$,$t'_\beta \in S$ and the $q'_{\beta'}$ are paths leaving $[1, m]$ to the right as in (7.11.1).

The PBW expansion of $g_{i_1} \cdots g_{i_m} \lambda_1 \cdots \lambda_m$ in terms of paths going first to the right is of the form

$$\pm[i_1 \cdots i_m | 1 \cdots m] + (\text{an } S\text{-linear combination of paths of positive length})$$

where $[i_1 \cdots i_m | 1 \cdots m]$ is the minor in X with columns $i_1, \ldots, i_m$.

Substituting this into (7.12.3) and looking at constant terms we obtain an identity in S:

$$\sum_{1\leqslant i_1<\cdots<i_m\leqslant n} \pm s'_{i_1\ldots i_m} [i_1 \cdots i_m | 1 \cdots m] = 0\,.$$

This is only possible if all $s'_{i_1\ldots i_m}$ are in $S_{>0}$, yielding a contradiction. □

7.13. *A Clifford action on the tilting bundle.* Let $\mathcal{T} = \bigoplus_a \mathcal{T}_a = \bigoplus_a p'^*\Omega^{a-1}(a)$ be the tilting bundle on $\mathcal{Z}$ defined in §6.3. In this section we construct an algebra morphism $C \longrightarrow \operatorname{End}_{\mathcal{Z}}(\mathcal{T})$ which we show to be an isomorphism afterwards. To construct the morphism it is sufficient (according to Lemma 7.4) to construct a left action of $\operatorname{Cliff}_S(q_\varphi)$ on $\mathcal{T}$.

We have to give the action of the generators. For the action of $F^\vee$ we use the composition

$$\partial\colon F^\vee \otimes \Omega^{b-1}(b) \longrightarrow \Omega^1(1)^\vee \otimes_{\mathbb{P}} \Omega^{b-1}(b) \longrightarrow \Omega^{b-2}(b-1)\,, \tag{7.13.1}$$

where the first map is obtained as the dual of the canonical map

$$\Omega^1(1) \longrightarrow F \otimes \mathcal{O}_{\mathbb{P}}$$

introduced for example in (3.5.1), while the second map is contraction.

For the G-action we use the composition

$$\begin{aligned}\theta\colon G \otimes p'^*\Omega^{b-1}(b) = q'^*\mathcal{G} \otimes_{\mathcal{O}_{\mathcal{Z}}} p'^*\Omega^{b-1}(b) &\longrightarrow p'^*\Omega^1(1) \otimes_{\mathcal{O}_{\mathcal{Z}}} p'^*\Omega^{b-1}(b)\\ &= p'^*(\Omega^1(1) \otimes_{\mathcal{O}_{\mathbb{P}}} \Omega^{b-1}(b)) \longrightarrow p'^*\Omega^b(b+1)\end{aligned} \tag{7.13.2}$$

where the first arrow is obtained from the description (5.2.1) and the second arrow is multiplication.

One checks that the $F^\vee$- and G-actions combine to give the requested action

$$\operatorname{Cliff}_S(q_\varphi) \otimes \mathcal{T} \longrightarrow \mathcal{T}\,.$$

Proposition 7.14. *The morphisms $C \longrightarrow E = \operatorname{End}_R(M)$ and $C \longrightarrow \operatorname{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T})$ defined in § 7.7 and § 7.13 are compatible with the isomorphism $\operatorname{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T}) \longrightarrow E$ of Theorem 6.5.*

Proof. From the construction in §7.7 we know that the constructed action $C_{ab} \otimes M_a \longrightarrow M_b$ lifts to an action

$$\operatorname{Cliff}_S(q_\varphi)_{b-a} \otimes \bigwedge\nolimits^a \mathcal{F} \longrightarrow \bigwedge\nolimits^b \mathcal{F}\,. \tag{7.14.1}$$

Likewise the same types of formulas show that the action $C_{ab} \otimes p'^*\Omega^{a-1}(a) \longrightarrow p'^*\Omega^{b-1}(b)$ lifts to an action

$$\operatorname{Cliff}_S(q_\varphi)_{b-a} \otimes q'^*\bigwedge\nolimits^a \mathcal{F} \longrightarrow q'^*\bigwedge\nolimits^b \mathcal{F}\,. \tag{7.14.2}$$

It is now easy to see the (7.14.1) and (7.14.2) are compatible, whence the originals are compatible by Theorem 6.2. $\square$

Theorem 7.15. *The map $C \longrightarrow \operatorname{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T})$ obtained by applying Lemma 7.4 to the action constructed in § 7.13 is an isomorphism.*

Proof. We have to show that $C_{ba} \longrightarrow \operatorname{Hom}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T}_b, \mathcal{T}_a)$ is an isomorphism. From Lemma 7.16 below together with the involution $C_{ba} \leftrightarrow C_{m+1-a,m+1-b}$ (see (7.5.1)), we easily deduce that we may assume $a + b \geqslant m + 1$. We make this assumption in the rest of the proof.

As S-modules we have

$$\operatorname{Cliff}_S(q_\varphi)_{a-b} = \bigoplus_c \bigwedge\nolimits^c G \otimes \bigwedge\nolimits^{b-a+c} F^\vee \otimes S = \bigoplus_c \bigwedge\nolimits^c \mathcal{G} \otimes \bigwedge\nolimits^{b-a+c} \mathcal{F}^\vee\,.$$

We equip $\operatorname{Cliff}_S(q_\varphi)_{a-b}$ with a filtration $\mathfrak{F}$ obtained from the value of c, that is,

$$\mathfrak{F}_u \operatorname{Cliff}_S(q_\varphi)_{a-b} = \bigoplus_{c=0}^{u} \bigwedge\nolimits^c G \otimes \bigwedge\nolimits^{b-a+c} F^\vee \otimes S = \bigoplus_{c=0}^{u} \bigwedge\nolimits^c \mathcal{G} \otimes \bigwedge\nolimits^{b-a+c} \mathcal{F}^\vee\,.$$

We will start by proving that the induced map

$$\mathfrak{F}_{m-b}\operatorname{Cliff}_S(q_\varphi)_{a-b} \longrightarrow \operatorname{Hom}_{\mathcal{O}_\mathcal{Z}}(p'^*\Omega^{b-1}(b), p'^*\Omega^{a-1}(a)) \tag{7.15.1}$$

is an epimorphism.

In §3.19 we have constructed an action by $\mathcal{O}_\mathbb{P}$-linear derivations

$$\partial\colon F^\vee \otimes \mathbb{K} \longrightarrow \mathbb{K}(-1)[1]\,. \tag{7.15.2}$$

This extends to an action by $\mathcal{O}_\mathcal{Z}$-linear derivations

$$\partial\colon F^\vee \otimes p'^*\mathbb{K} \longrightarrow p'^*\mathbb{K}(-1)[1]\,.$$

We produce an additional action

$$\theta\colon G \otimes p'^*\mathbb{K} \longrightarrow p'^*\mathbb{K}(1)[-1] \tag{7.15.3}$$

by $p'^*\mathbb{K}$-linearly extending the $\mathbb{K}$-linear map

$$\begin{aligned} G \otimes \mathbb{K} &\longrightarrow G \otimes \Omega(1)^\vee \otimes_{\mathcal{O}_\mathbb{P}} \Omega(1) \otimes_\mathbb{P} \mathbb{K} \\ &\longrightarrow G \otimes \Omega(1)^\vee \otimes_{\mathcal{O}_\mathbb{P}} \mathbb{K}(1)[-1] \\ &\subset \mathcal{O}_\mathcal{Z} \otimes_{\mathcal{O}_\mathbb{P}} \mathbb{K}(1)[-1] \qquad\qquad (3.5.1) \\ &= p'^*\mathbb{K}(1)[-1] \end{aligned}$$

where the second arrow is multiplication in the graded sheaf of algebras $\mathbb{K}$ via the inclusion $\Omega^1 \subset \mathbb{K}^{-1}$. Since the image of this inclusion consists of closed elements the resulting multiplication is compatible with the differential. (Note: the multiplication $F \otimes \mathbb{K} \longrightarrow \mathbb{K}(1)[-1]$ is *not* compatible with the differential.)

One readily checks that (7.15.2) and (7.15.3) combine to give an $\mathcal{O}_\mathcal{Z}$-linear action

$$\operatorname{Cliff}_S(q_\varphi)_s \otimes p'^*\mathbb{K} \longrightarrow p'^*\mathbb{K}(s)[-s]\,. \tag{7.15.4}$$

Put $s = a - b$. We obtain an action

$$\operatorname{Cliff}_S(q_\varphi)_{a-b} \otimes p'^*\mathbb{K}(b)[-b+1] \longrightarrow p'^*\mathbb{K}(a)[-a+1]$$

which after truncating in homological degree zero becomes

$$\operatorname{Cliff}_S(q_\varphi)_{a-b} \otimes p'^*\mathbb{K}_{\leqslant b-1}(b) \longrightarrow p'^*\mathbb{K}_{\leqslant a-1}(a)$$

so that we finally get a composition

$$\begin{aligned} \mathfrak{F}_{m-b}\operatorname{Cliff}_S(q_\varphi)_{a-b} &\hookrightarrow \operatorname{Cliff}_S(q_\varphi)_{a-b} \longrightarrow \\ &\operatorname{RHom}_{\mathcal{D}(\mathcal{Z})}(p'^*\mathbb{K}_{\leqslant b-1}(b), p'^*\mathbb{K}_{\leqslant a-1}(a)) \cong \operatorname{Hom}_{\mathcal{O}_\mathcal{Z}}(\mathcal{T}_b, \mathcal{T}_a)\,. \end{aligned} \tag{7.15.5}$$

It is easy to check that the second map coincides with the one obtained from our action of C on $\mathcal{T}$. We will show that (7.15.5) is an epimorphism.

Using the same methods as above we may define $\mathcal{O}_\mathcal{Y}$-linear actions

$$\begin{aligned} \partial\colon F^\vee \otimes p^*\mathbb{K} &\longrightarrow p^*\mathbb{K}(-1)[1] \\ \theta\colon G \otimes p^*\mathbb{K} &\longrightarrow p^*\mathbb{K}(1)[-1] \end{aligned}$$

which are compatible with the natural map $p^*\mathbb{K} \longrightarrow j_*p'^*\mathbb{K}$. For example θ is obtained by extending

$$\begin{aligned} G \otimes \mathbb{K} &\longrightarrow G \otimes F^\vee \otimes_K F \otimes_K \mathbb{K} \\ &\longrightarrow G \otimes F^\vee \otimes \mathbb{K}(1)[-1] \\ &\subset S \otimes_\mathbb{P} \mathbb{K}(1)[-1] \\ &= p^*\mathbb{K}(1)[-1]\,. \end{aligned}$$

Unfortunately θ is now not compatible with the differential. However the commutator

$$d_{\mathbb{K}}\theta + \theta d_{\mathbb{K}} \colon G \otimes p^*\mathbb{K} \longrightarrow p^*\mathbb{K}(1)$$

is given by multiplication with the cosection $\Phi \colon G \longrightarrow \mathcal{O}_{\mathcal{Y}}(1)$ defined in (5.2.2). Written compactly,

$$d_{\mathbb{K}}\theta + \theta d_{\mathbb{K}} = \Phi\,.$$

Let $\mathbb{L} = (\bigwedge_{\mathcal{Y}}(q^*\mathcal{G} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathcal{O}_{\mathbb{P}}(-1)), \partial_{\Phi(-1)})$ be the Koszul complex of locally free $\mathcal{O}_{\mathcal{Y}}$-modules resolving $j_*\mathcal{O}_{\mathcal{Z}}$ which was introduced in (5.2.3). Multiplication by elements of G defines an action

$$\widetilde{\theta} \colon G \otimes \mathbb{L} \longrightarrow \mathbb{L}(1)[-1]$$

which is again is not compatible with the differential. However one computes

$$d_{\mathbb{L}}\widetilde{\theta} + \widetilde{\theta} d_{\mathbb{L}} = \Phi$$

so that the combined actions

$$\partial \overset{\mathrm{def}}{=} \partial_{13} \colon F^\vee \otimes (\mathbb{L} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K}) \longrightarrow \mathbb{L} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K}(-1)[1]$$

$$\Theta \overset{\mathrm{def}}{=} \pm\widetilde{\theta}_{12} \otimes 1 + 1 \otimes \theta_{13} \colon G \otimes (\mathbb{L} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K}) \longrightarrow \mathbb{L} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K}(1)[-1]$$

commute with the total differential on the complex associated to the double complex $\mathbb{L} \otimes_{\mathcal{O}_{\mathbb{P}}} p^*\mathbb{K}$. (Here the subscripts indicate the factors of the tensor product to which the maps apply.)

It is easy to see that these actions combine to give an action

$$\operatorname{Cliff}_S(q_\varphi)_s \otimes (\mathbb{L} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K}) \longrightarrow \mathbb{L} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K}(s)[-s] \tag{7.15.6}$$

which is compatible with the total differential and with the natural map

$$\mathbb{L} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K} \longrightarrow j_*\mathcal{O}_{\mathcal{Z}} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K} = j_*p'^*\mathbb{K}\,.$$

Put $s = a - b$. Then (7.15.6) restricts to a map

$$\operatorname{Cliff}_S(q_\varphi)_{a-b} \otimes p^*\mathbb{K}_{\leqslant b-1}(b) \longrightarrow \mathbb{L} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K}_{\leqslant a-1}(a) \tag{7.15.7}$$

For $t \in \mathbb{N}$ and C a complex let $\sigma^{\geqslant t}C$ denote the naive truncation of C in cohomological degrees $\geqslant t$. Then (7.15.7) restricts again to

$$\mathfrak{F}_{m-b}\operatorname{Cliff}_S(q_\varphi)_{a-b} \otimes p^*\mathbb{K}_{\leqslant b-1}(b) \longrightarrow \sigma^{\geqslant -(m-b)}\mathbb{L} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K}_{\leqslant a-1}(a)$$

We now obtain a commutative diagram

$$\begin{tikzcd}
\mathfrak{F}_{m-b}\operatorname{Cliff}_S(q_\varphi)_{a-b} \arrow[r, "\alpha"] \arrow[dd, hook, "\beta"'] & \operatorname{Hom}_{\mathcal{O}_{\mathcal{Y}}}(p^*\mathbb{K}_{\leqslant b-1}(b), \sigma^{\geqslant -(m-b)}\mathbb{L} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K}_{\leqslant a-1}(a)) \arrow[d, "\gamma_1"] \\
 & \operatorname{RHom}_{\mathcal{D}(\mathcal{Y})}(p^*\mathbb{K}_{\leqslant b-1}(b), \sigma^{\geqslant -(m-b)}\mathbb{L} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K}_{\leqslant a-1}(a)) \arrow[d, "\gamma_2"] \\
\operatorname{Cliff}_S(q_\varphi)_{a-b} \arrow[r] \arrow[d, equal] & \operatorname{RHom}_{\mathcal{D}(\mathcal{Y})}(p^*\mathbb{K}_{\leqslant b-1}(b), \mathbb{L} \otimes_{\mathcal{O}_{\mathcal{Y}}} p^*\mathbb{K}_{\leqslant a-1}(a)) \arrow[d, "\delta"] \\
\operatorname{Cliff}_S(q_\varphi)_{a-b} \arrow[r, "\epsilon"'] & \operatorname{RHom}_{\mathcal{D}(\mathcal{Z})}(p'^*\mathbb{K}_{\leqslant b-1}(b), p'^*\mathbb{K}_{\leqslant a-1}(a))
\end{tikzcd}$$

where the horizontal arrows are obtained from the Clifford algebra actions and the vertical arrows are the natural ones. The commutativity of the lower square follows from the above discussion.

Looking back at (7.15.5) we see that we have to show that $\epsilon\beta$ is an epimorphism on degree zero cohomology. So we have to show that $\delta\gamma_2\gamma_1\alpha$ is an epimorphism on degree zero cohomology.

The fact that $\mathbb{L}$ is a resolution of $j_*\mathcal{O}_\mathcal{Z}$ and formal adjointness arguments imply that δ is a quasi-isomorphism (in fact this is the basis of the proof of Theorem 5.3).

We claim that γ_2 is an epimorphism on degree zero cohomology. To this end we look at the distinguished triangle

$$\sigma^{\geqslant -(m-b)}\mathbb{L}\otimes_{\mathcal{O}_\mathcal{Y}} p^*\mathbb{K}_{\leqslant a-1}(a)) \to \mathbb{L}\otimes_{\mathcal{O}_\mathcal{Y}} p^*\mathbb{K}_{\leqslant a-1}(a)) \to \sigma^{<-(m-b)}\mathbb{L}\otimes_{\mathcal{O}_\mathcal{Y}} p^*\mathbb{K}_{\leqslant a-1}(a)) \to$$

It is sufficient to prove that

$$\operatorname{Hom}_{\mathcal{D}(\mathcal{Y})}\left(p^*\mathbb{K}_{\leqslant b-1}(b), \sigma^{<-(m-b)}\mathbb{L}\otimes_{\mathcal{O}_\mathcal{Y}} p^*\mathbb{K}_{\leqslant a-1}(a)\right) = 0$$

which in turn follows from

$$\operatorname{Hom}_{\mathcal{D}(\mathcal{Y})}\left(p^*\mathbb{K}_{\leqslant b-1}(b), \mathbb{L}^{-c}[c]\otimes_{\mathcal{O}_\mathcal{Y}} p^*\mathbb{K}_{\leqslant a-1}(a)\right) = 0$$

for $c > m-b$. To prove this last equation we note that

$$\operatorname{Hom}_{\mathcal{D}(\mathcal{Y})}\left(p^*\mathbb{K}_{\leqslant b-1}(b), \mathbb{L}^{-c}[c]\otimes_{\mathcal{O}_\mathcal{Y}} p^*\mathbb{K}_{\leqslant a-1}(a)\right) = \bigwedge^c G\otimes H^c(\mathbb{P}, \mathcal{M}^b_a(-c))\otimes S$$

The required vanishing now follows from Theorem 3.9(2,4,5).

We also claim that γ_1 is a quasi-isomorphism. This follows immediately from §3.12(e) which states that the sheaf-Homs between the terms of $p^*\mathbb{K}_{\leqslant b-1}(b)$ and $\sigma^{\geqslant(m-b)}\mathbb{L}\otimes_{\mathcal{O}_\mathcal{Y}} p^*\mathbb{K}_{\leqslant a-1}(a)$ have no higher cohomology.

Finally we claim that α is a quasi-isomorphism. To prove this we filter the complex $\operatorname{Hom}_{\mathcal{O}_\mathcal{Y}}(p^*\mathbb{K}_{\leqslant b-1}(b), \sigma^{\geqslant -(m-b)}\mathbb{L}\otimes_{\mathcal{O}_\mathcal{Y}} p^*\mathbb{K}_{\leqslant a-1}(a))$ by the degrees in the $\mathbb{L}$-complex and we equip $\operatorname{Cliff}_S(q_\varphi)$ with the filtration $\mathfrak{F}$ defined above.

Taking associated graded complexes we find that we have to show that

$$\begin{aligned}\bigwedge^c G\otimes\bigwedge^{a-b+c}F^\vee\otimes S &\longrightarrow \operatorname{Hom}_{\mathcal{O}_\mathcal{Y}}\left(p^*\mathbb{K}_{\leqslant b-1}(b), \mathbb{L}^{-c}[c]\otimes p^*\mathbb{K}_{\leqslant a-1}(a)\right)\\ &= \bigwedge^c G\otimes\operatorname{Hom}_{\mathcal{O}_\mathbb{P}}\left(\mathbb{K}_{\leqslant b-1}(b), \mathbb{K}_{\leqslant a-1}(a)(-c)[c]\right)\otimes S\end{aligned}$$

is a quasi-isomorphism for $c \leqslant m-b$. One verifies that up to sign this is in fact the map $\mathrm{id}\otimes\partial\otimes\mathrm{id}$ where ∂ is as defined in (3.20.1). To finish the proof that (7.15.5) is an epimorphism it is now sufficient to invoke Lemma 3.20.

At this point we know that $C_{ba} \longrightarrow \operatorname{Hom}_{\mathcal{O}_\mathcal{Z}}(\mathcal{T}_b, \mathcal{T}_a)$ is an epimorphism. We will proceed to show that it is an isomorphism. With notations as in Lemma 7.12 (swapping a and b) we may construct a commutative diagram

(7.15.8)
$$\begin{tikzcd} P_1\oplus Q \arrow[r, "\rho"] \arrow[d, "\exists"', "(\alpha_1,\alpha_2)"] & P_0 \arrow[r] \arrow[d, equal] & C_{ba} \arrow[r] \arrow[d, two heads] & 0 \\ P_1 \arrow[r] & P_0 \arrow[r] & \operatorname{Hom}_\mathcal{Z}(\mathcal{T}_b, \mathcal{T}_a) \arrow[r] & 0 \end{tikzcd}$$

where $P_0 = \mathfrak{F}_{m-b}\operatorname{Cliff}_S(q_\varphi)_{a-b}$ is as in (7.15.1). The upper exact sequence is obtained from Lemma 7.12. The arrow $P_0 \to \operatorname{Hom}_\mathcal{Z}(\mathcal{T}_b, \mathcal{T}_a)$ is defined as the composition $P_0 \to C_{ba} \to \operatorname{Hom}_\mathcal{Z}(\mathcal{T}_b, \mathcal{T}_a)$. By the second and third row of (5.3.1) with $c = 0$ (also using the assumption $a+b \geqslant m+1$) we know the minimal resolution of $\operatorname{Hom}_\mathcal{Z}(\mathcal{T}_b, \mathcal{T}_a)$, which tells us that we can complete the lower row as we did.

Then the existence of (α_1, α_2) follows but its properties are a priori unknown. Nonetheless we claim that α_1 must be an isomorphism. Assume this is not the case. Choose two sets of homogeneous bases $(x_i)_{i=1,\ldots,N}$, $(y_i)_{i=1,\ldots,N}$ for P_1 ordered in ascending degree. Let A be the matrix of α_1 with respect to these bases. Since A is not invertible, easy degree considerations show that after change of basis A may be put in the form

$$\begin{pmatrix} 1 & \cdots & 0 & 0 & A_{1t+1} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & A_{t,t+1} & \cdots & A_{tN} \\ 0 & \cdots & 0 & 0 & A_{t+1t+1} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & A_{N,t+1} & \cdots & A_{NN} \end{pmatrix}.$$

It follows that P_1, as a graded S-module, may be decomposed as $P_1 = P_1' \oplus P_1''$ with $P_1'' \cong S(-u)$ for $u = \deg x_t$ such that the restriction of α_1 to P_1'' is zero. It then follows from (7.15.8) that $\rho \mid_{P_1''} = 0$ as well. In other words $P_1'' \subseteq \ker(\rho \mid_{P_1})$. Since $P_1'' \not\subseteq S_{>0}P_1$ this contradicts Lemma 7.12.

Hence α_1 is an isomorphism and as a result (α_1, α_2) is an epimorphism. Then diagram (7.15.8) easily yields that $C_{ba} \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T}_b, \mathcal{T}_a)$ is an isomorphism. $\square$

The following lemma was used.

Lemma 7.16. *There is a commutative diagram*

$$\begin{tikzcd} \mathrm{Cliff}_S(q_\varphi)_{a-b} \arrow[r] \arrow[d, "\alpha"'] & q'_* p'^* \mathcal{M}_a^b \arrow[d, "\beta"] \\ \mathrm{Cliff}_S(q_\varphi)_{a-b} \arrow[r] & q'_* p'^* \mathcal{M}_{m+1-b}^{m+1-a} \end{tikzcd}$$

where the horizontal maps are those in Theorem 7.15, α is obtained from the involution on $\mathrm{Cliff}_S(q_\varphi)$ which is the identity on $\mathcal{F}^\vee \oplus \mathcal{G}$, and β is obtained from the isomorphism $\mathcal{M}_a^b \cong \mathcal{M}_{m+1-b}^{m+1-a}$ exhibited in Lemma 3.8.1.

Proof. The isomorphism $\mathcal{M}_a^b \cong \mathcal{M}_{m+1-b}^{m+1-a}$ in Lemma 3.8.1 is derived from the non-degenerate pairing (3.8.3)

$$- \wedge -\colon \Omega^{a-1}(a) \otimes_{\mathcal{O}_{\mathbb{P}}} \Omega^{m-a}(-a) \longrightarrow \Omega^{m-1}$$

and likewise the induced isomorphism $p'^*\mathcal{M}_a^b \cong p'^*\mathcal{M}_{m+1-b}^{m+1-a}$ can be obtained from the induced pairing

$$- \wedge -\colon p'^*\Omega^{a-1}(a) \otimes_{\mathcal{O}_{\mathbb{P}}} p'^*\Omega^{m-a}(-a) \longrightarrow p'^*\Omega^{m-1}$$

It is therefore sufficient to show that for $\lambda \in F^\vee$ and $g \in G$ the actions of ∂_λ and θ_g as defined in §7.13 are self-adjoint for this pairing. This is an easy exercise which we leave to the reader. $\square$

Combining the above theorem with Proposition 7.14 we have the main result of this section.

Theorem 7.17. *The endomorphism algebra $E = \mathrm{End}_R(M)$ is isomorphic to the quiverized Clifford algebra C.*

8. The Commutative Desingularization as a Moduli Space

Having completed the proofs of the statements contained in Theorems A-C in the Introduction we now include some miscellaneous sections. In this section we show that the canonical commutative desingularization $\mathcal{Z}$ of $\operatorname{Spec} R$ can be obtained as a fine moduli space for certain representations over the non-commutative one.

Specifically, we prove in Theorem 8.9 that $\mathcal{Z}$ represents the functor of flat families of representations W of $\widetilde{\mathsf{Q}}$ which have dimension vector $(1, m-1, \binom{m-1}{2}, \dots, 1)$ and which are generated by W_m. We then identify the points in $\mathcal{Z}$ corresponding to the simple representations W as those lying over the non-singular locus of $\operatorname{Spec} R$.

8.1. *Quiver representations.* A K-representation, V, of a quiver Γ associates a (finite-dimensional) K-vector space V_i to each vertex i of Γ and a linear map $V(a)\colon V(i) \longrightarrow V(j)$ for each arrow $a\colon i \longrightarrow j$. A homomorphism f of representations from V to V' is given by a collections of linear maps for each vertex $f(i)\colon V(i) \longrightarrow V'(i)$ so that the obvious diagram commutes. The category $\mathfrak{rep}(\Gamma)$ of representations is an abelian category. The dimension vector of V, a function from the vertices of Γ to the natural numbers, assigns to i the K-rank of $V(i)$. The representations of Γ with a fixed dimension vector $\theta = (\theta(i))_i$ are parametrized by the vector space $\prod_{i \longrightarrow j} \operatorname{Hom}_K(V(i), V(j))$, and thus the isomorphism classes of representations V with dimension vector θ are in one-one correspondence with the orbits under the action of $\prod_i \mathrm{GL}_{\theta(i)}(K)$.

These notions clearly generalize to the case where K is an arbitrary commutative ring and each $V(i)$ is a free K-module of finite rank.

8.2. *Baby case.* The Beĭlinson algebra associated to a vector space F of rank m over the field K is the order-m quiverization (see §8.1) $Q_m(\bigwedge F^\vee)$ of the exterior algebra of $F^\vee$.

The Beĭlinson algebra can be represented as the path algebra of the Beĭlinson quiver

$$\mathsf{Q}:\qquad 1 \;\substack{\xleftarrow{\quad\lambda_m\quad} \\ \vdots \\ \xleftarrow{\quad\lambda_1\quad}}\; 2 \;\substack{\xleftarrow{\quad\lambda_m\quad} \\ \vdots \\ \xleftarrow{\quad\lambda_1\quad}}\; \cdots\cdots \;\substack{\xleftarrow{\quad\lambda_m\quad} \\ \vdots \\ \xleftarrow{\quad\lambda_1\quad}}\; m$$

equipped with the anti-commutativity relations $\lambda_i\lambda_j + \lambda_j\lambda_i = 0 = \lambda_i^2$. The category $\mathfrak{rep}(\mathsf{Q})$ is equivalent to the category of graded left $\bigwedge F^\vee$-modules with support in degrees $1, \dots, m$ (see Lemma 7.4).

For an arbitrary commutative K-algebra A we let $\mathcal{R}(A)$ be the set of isomorphism classes W of representations of Q of the form

$$W:\qquad W_1 \;\substack{\xleftarrow{\quad\lambda_m\quad} \\ \vdots \\ \xleftarrow{\quad\lambda_1\quad}}\; W_2 \;\substack{\xleftarrow{\quad\lambda_m\quad} \\ \vdots \\ \xleftarrow{\quad\lambda_1\quad}}\; \cdots\cdots \;\substack{\xleftarrow{\quad\lambda_m\quad} \\ \vdots \\ \xleftarrow{\quad\lambda_1\quad}}\; W_m$$

such that each W_a is a projective A-module of rank $\binom{m-1}{a-1}$, and W is generated by $W_m = A$.

For a projective A-module P of rank $m-1$ and a *split monomorphism* $\alpha\colon P \longrightarrow F \otimes A$, define a representation $W_\alpha \in \mathcal{R}(A)$ by

$$(W_\alpha)_a = \bigwedge\nolimits_A^{m-a} P^\vee$$

for $a = 1, \dots, m$, with $P^\vee = \operatorname{Hom}_A(P, A)$. Define the action of $\lambda \in F^\vee$ on W_α by the left exterior multiplication

$$\alpha^\vee(\lambda) \wedge -\colon \bigwedge\nolimits_A^{m-a} P^\vee \longrightarrow \bigwedge\nolimits_A^{m-a+1} P^\vee\,,$$

where $\alpha^\vee \colon F^\vee \otimes A \twoheadrightarrow P^\vee$ is the A-dual of α.

Lemma 8.3. *Every $W \in \mathcal{R}(A)$ is of the form W_α for a uniquely determined A-projective P of rank $m-1$ and split monomorphism $\alpha\colon P \longrightarrow F \otimes A$.*

Proof. Let $W \in \mathcal{R}(A)$. Viewed as a left module over $(\bigwedge F^\vee) \otimes A = \bigwedge_A(F^\vee \otimes A)$, W is generated by $W_m = A$. This gives in particular a surjective homomorphism

$$\pi\colon F^\vee \otimes A \longrightarrow W_{m-1}\,.$$

If $W = W_\alpha$ then $W_{m-1} = \bigwedge_A^1 P^\vee = P^\vee$, and thus $\alpha = \pi^\vee$ can be reconstructed from W, giving uniqueness.

For W arbitrary, put $I = \ker \pi$. As W is generated by W_m we find that W is a quotient of $\bigwedge_A((F^\vee \otimes A)/I) = \bigwedge_A W_{m-1}$. Since W and $\bigwedge_A W_{m-1}$ have the same rank, we see that $W \cong \bigwedge_A W_{m-1}$ is of the form W_α. $\square$

With $\mathbb{P} = \mathbb{P}(F^\vee)$ once more the projective space of linear forms, let $\mathcal{U} = \Omega^1(1) = \ker(F \otimes \mathcal{O}_\mathbb{P} \longrightarrow \mathcal{O}_\mathbb{P}(1))$ be the tautological bundle. Any split monomorphism $P \longrightarrow F \otimes A$ with P of rank $m-1$ is uniquely obtained as a pullback of $\mathcal{U} \longrightarrow F \otimes \mathcal{O}_\mathbb{P}$ across an A-point $\eta\colon \operatorname{Spec} A \longrightarrow \mathbb{P}$ of $\mathbb{P}$. Combining this with Lemma 8.3 we obtain the following corollary.

Corollary 8.4. *The functor $\mathcal{R}$ is representable by $\mathbb{P}(F^\vee)$; equivalently, $\mathbb{P}(F^\vee)$ is a fine moduli space for $\mathcal{R}$. The universal bundle is given by $\mathcal{B}_0 = \bigwedge_{\mathbb{P}(F^\vee)} \mathcal{U}^\vee$, where $\lambda \in F^\vee$ acts via $\partial^\vee(\lambda) \wedge -$.* $\square$

8.5. *Representations of the quiverized Clifford algebra.* Reintroduce now the second K-vector space G of rank n, with its fixed basis $\{g_1, \dots, g_n\}$, and consider again from §7.5 the doubled Beĭlinson quiver on $F^\vee$ and G

$$\widetilde{\mathsf{Q}}: \quad \begin{tikzcd} 1 \arrow[r, shift right=1.5ex, "g_1 \,\cdots\, g_n"'] & 2 \arrow[l, shift right=1.5ex, "\lambda_1 \,\cdots\, \lambda_m"'] \arrow[r, shift right=1.5ex, "g_1 \,\cdots\, g_n"'] & \cdots\cdots \arrow[l, shift right=1.5ex, "\lambda_1 \,\cdots\, \lambda_m"'] \arrow[r, shift right=1.5ex, "g_1 \,\cdots\, g_n"'] & m \arrow[l, shift right=1.5ex, "\lambda_1 \,\cdots\, \lambda_m"'] \end{tikzcd}$$

with relations as before. Again let C be its path algebra.

For an arbitrary commutative K-algebra A, let $\widetilde{\mathcal{R}}(A)$ consist of those isomorphism classes of representations

$$W: \quad \begin{tikzcd} W_1 \arrow[r, shift right=1.5ex, "g_1 \,\cdots\, g_n"'] & W_2 \arrow[l, shift right=1.5ex, "\lambda_1 \,\cdots\, \lambda_m"'] \arrow[r, shift right=1.5ex, "g_1 \,\cdots\, g_n"'] & \cdots\cdots \arrow[l, shift right=1.5ex, "\lambda_1 \,\cdots\, \lambda_m"'] \arrow[r, shift right=1.5ex, "g_1 \,\cdots\, g_n"'] & W_m \arrow[l, shift right=1.5ex, "\lambda_1 \,\cdots\, \lambda_m"'] \end{tikzcd}$$

such that each W_a is a projective A-module of rank $\binom{m-1}{a-1}$, and W is generated as a left C-module by $W_m = A$.

Proposition 8.6. *Let $W \in \widetilde{\mathcal{R}}(A)$. Then the central elements $x_{ij} \in C$ act as scalars (elements of A) on W. Furthermore, W is generated by W_m as a left module over $\bigwedge_A(F^\vee \otimes A)$.*

Proof. Each homogeneous $A \otimes C$-linear endomorphism of W is determined by its action on W_m. From the fact that $W_m = A$, we deduce that every such endomorphism is given by multiplication by some element of A. In particular, this holds for multiplication by x_{ij}.

Any element of C can be written as a linear combination of products $e_b\boldsymbol{\lambda g x}e_a$, where $\boldsymbol{\lambda}$, $\boldsymbol{g}$, and $\boldsymbol{x}$ are products of λ_k, g_l, and x_{ij}. As each g_l acts with degree $+1$, $g_l W_m = 0$. It follows that W is generated by W_m over $(\bigwedge F^\vee)\otimes A = \bigwedge_A(F^\vee \otimes A)$ alone. $\square$

8.7. Suppose now we are given a projective A-module P of rank $m-1$, and a pair of homomorphisms

$$\alpha\colon P \longrightarrow F\otimes A\,, \qquad \beta\colon P^\vee \longrightarrow G^\vee \otimes A$$

with α a split monomorphism. Define $W_{\alpha\beta}\in\widetilde{\mathcal{R}}(A)$ to have

$$(W_{\alpha\beta})_a = \textstyle\bigwedge_A^{m-a} P^\vee$$

as before, with $\lambda\in F^\vee$ again acting via $\alpha^\vee(\lambda)\wedge -$, and with $g\in G$ acting via the contraction $\beta^\vee(g)\curlyvee -$. Explicitly, g sends $u^1\wedge\cdots\wedge u^{m-a}$ to

$$\sum_{j=1}^{m-a}(-1)^{j-1}u^j(\beta^\vee(g))\left(u^1\wedge\cdots\wedge\widehat{u^j}\wedge\cdots\wedge u^{m-a}\right) \quad \in \textstyle\bigwedge_A^{m-a-1}P^\vee\,.$$

Proposition 8.8. *Every $W\in\widetilde{\mathcal{R}}(A)$ is of the form $W_{\alpha\beta}$ for a uniquely determined projective P of rank $m-1$ and a pair of homomorphisms*

$$\alpha\colon P \longrightarrow F\otimes A \quad , \quad \beta\colon P^\vee \longrightarrow G^\vee\otimes A$$

with α a split monomorphism.

Proof. Any representation class $W\in\widetilde{\mathcal{R}}(A)$ can be viewed as an object of $\mathcal{R}(A)$ by simply ignoring the rightward-pointing arrows of $\widetilde{\mathsf{Q}}$. By Lemma 8.3, such an object is necessarily of the form $W_\alpha = \bigwedge_A P^\vee$ for some P and some monomorphism $\alpha\colon P\hookrightarrow F\otimes A$. It remains only to construct β.

The central elements $x_{ij} = \lambda_i g_j + g_j\lambda_i \in C$ act on W as multiplication by certain scalars $a_{ij}\in A$. Applying this to the generator $1\in A = W_m$, we obtain

$$a_{ij} = g_j\lambda_i\,,$$

so that in particular each g_j acts as the left super-S-derivation on $\bigwedge_A P^\vee$ sending $\alpha^\vee(\lambda_i)$ to a_{ij}. Hence the action of G on $\bigwedge_A P^\vee$ is provided by a homomorphism $\gamma\colon G\otimes A \longrightarrow P$, which dualizes to a map $\beta\colon P^\vee \longrightarrow G^\vee\otimes A$ such that $\gamma(g)$ is given by contraction with $\beta^\vee(g)$ for each $g\in G$. This shows that $W\cong W_{\alpha\beta}$. As in Lemma 8.3, both α and β can be reconstructed from W. $\square$

Let $\mathcal{Z}$ again be the Springer desingularization of $\operatorname{Spec} R$. As in §5.2, we write

$$\mathcal{Z} = \underline{\operatorname{Spec}}\left(\operatorname{Sym}_{\mathbb{P}(F^\vee)}(\mathcal{U}^\vee\otimes G)\right)\,.$$

The bundle $\bigwedge_{\mathbb{P}}\mathcal{U}^\vee\otimes_{\mathbb{P}}\operatorname{Sym}_{\mathbb{P}}(\mathcal{U}^\vee\otimes G)$ carries a natural C-action where $\lambda\in F^\vee$ acts via $\partial^\vee(\lambda)\wedge -$ and $g\in G$ sends a section e of $\mathcal{U}^\vee$ to $e\otimes g$ and fixes $\mathcal{U}^\vee\otimes G$. Denote this latter super-derivation by $g\curlyvee -$. Letting $\mathcal{B}$ be the $\mathcal{O}_{\mathcal{Z}}$-module determined by $\bigwedge_{\mathbb{P}}\mathcal{U}^\vee\otimes_{\mathbb{P}}\operatorname{Sym}_{\mathbb{P}}(\mathcal{U}^\vee\otimes G)$, we see that $\mathcal{B} = p'^*\mathcal{B}_0$, where $\mathcal{B}_0 = \bigwedge_{\mathbb{P}}\mathcal{U}^\vee$ is as in Corollary 8.4 and $p'\colon\mathcal{Z}\longrightarrow\mathbb{P}$ is the projection. Of course $\mathcal{B}$ is still a C-module.

Theorem 8.9. *The functor $\widetilde{\mathcal{R}}$ is representable by $\mathcal{Z}$. The universal bundle is given by $\mathcal{B} = p'^*(\bigwedge_{\mathbb{P}}\mathcal{U}^\vee)$.*

Proof. An A-point of $\mathcal{Z}$ consists of two pieces of data. The first of these is a point $\eta\colon \operatorname{Spec} A \longrightarrow \mathbb{P}$, and we obtain from the canonical map $\mathcal{U} \longrightarrow F \otimes \mathcal{O}_{\mathbb{P}}$ a split monomorphism

$$\partial_\eta \colon \mathcal{U}_\eta \longrightarrow F \otimes A$$

with $\mathcal{U}_\eta$ an A-projective of rank $m-1$. The other information carried by a point of $\mathcal{Z}$ is an A-point $\xi\colon \operatorname{Spec} A \longrightarrow \underline{\operatorname{Spec}} \operatorname{Sym}_{\mathbb{P}}(\mathcal{U}_\eta^\vee \otimes G)$. Such a point corresponds to an A-linear map $\mathcal{U}_\eta^\vee \otimes G \longrightarrow A$, which by adjunction yields a homomorphism $\beta\colon \mathcal{U}_\eta^\vee \longrightarrow G^\vee \otimes A$.

Thus the A-points of $\mathcal{Z}$ are in one-one correspondence with the pairs (α, β), i.e., with the elements of $\widetilde{\mathcal{R}}(A)$. This proves that $\mathcal{Z}$ represents $\widetilde{\mathcal{R}}$. It is easy to see that the induced actions of $F^\vee$ and G on $\mathcal{B}_\xi = (\mathcal{B}_0)_\eta$ define an isomorphism $\mathcal{B}_\xi \cong W_{\alpha\beta}$. □

8.10. *Simple representations.* Our next task is to identify the points of $\mathcal{Z}$ corresponding to the simple representations $W \in \widetilde{\mathcal{R}}(A)$. We shall see that they are precisely those points lying over the non-singular locus of $\operatorname{Spec} R$. We first record an easy lemma.

Lemma 8.11. *Assume that $A = K$. Then $W \in \widetilde{\mathcal{R}}(K)$ is simple if and only if W is generated by W_1.*

Proof. If we consider only the action of the λ_i then $W = \bigwedge P^\vee$. We see that any subrepresentation of W contains its socle $W_m = \bigwedge_A^{m-1} P^\vee$. Hence if W_1 generates W then this subrepresentation must be everything. □

Lemma 8.12. *The following are equivalent for $W = W_{\alpha\beta} \in \widetilde{\mathcal{R}}(K)$:*

(1) W is a simple left C-module;
(2) $\beta\colon P^\vee \longrightarrow G^\vee$ is a monomorphism.

Proof. The perfect pairing

$$\bigwedge^{m-a} P^\vee \times \bigwedge^{a-1} P^\vee \longrightarrow \bigwedge^{m-1} P^\vee \cong A$$

defines an isomorphism

$$W_a = \bigwedge\nolimits_A^{m-a} P^\vee \cong \left(\bigwedge\nolimits_A^{a-1} P^\vee\right)^\vee \cong \bigwedge^{a-1} P\,.$$

For any $g \in G$, then, the diagram

$$\begin{tikzcd}
\bigwedge^{m-a} P^\vee \arrow[r, "\cong"] \arrow[d, "\beta^\vee(g)\curlyvee -"'] & \bigwedge^{a-1} P \arrow[d, "\beta^\vee(g)\wedge -"] \\
\bigwedge^{m-a-1} P^\vee \arrow[r, "\cong"] & \bigwedge^{a} P
\end{tikzcd}$$

is commutative. We see from Lemma 8.11 that W is generated by W_1 if and only if

$$\beta^\vee(g) \curlyvee - \colon \bigwedge^{m-1} P^\vee \otimes G^\vee \longrightarrow \bigwedge^{m-2} P^\vee$$

is surjective, if and only if $\beta\colon P^\vee \longrightarrow G^\vee$ is injective. □

Proposition 8.13. *A representation $W_{\alpha\beta} \in \widetilde{\mathcal{R}}(K)$ is simple if and only if the corresponding point in $\mathcal{Z}$ lies over the non-singular locus of* $\operatorname{Spec} R$.

Proof. Recall that the projection $q'\colon \mathcal{Z} \longrightarrow \operatorname{Spec} R$ is an isomorphism over the non-singular locus of $\operatorname{Spec} R$. One checks that the composition $\mathcal{Z} \xrightarrow{q'} \operatorname{Spec} R \hookrightarrow \operatorname{Spec} S \cong F \otimes G^\vee$ sends a point of $\mathcal{Z}$, viewed as a pair of homomorphisms (α, β) as above, to the composition

$$K \longrightarrow P \otimes P^\vee \xrightarrow{\alpha\otimes\beta} F \otimes G^\vee .$$

Thus a point of $\mathcal{Z}$ corresponds to a simple C-module if, and only if, $\alpha \otimes \beta$ has rank $n-1$, which occurs exactly when it lies over the non-singular locus of $\operatorname{Spec} R$. $\square$

9. Explicit Minimal Presentations

In this section we will write down explicit minimal S-presentations for the Cohen-Macaulay modules $\operatorname{Hom}_R(M_a, M_b)$. By Theorem 7.17 this amounts to giving an S-free presentation of C_{ab}. By the involution $C_{ab} \leftrightarrow C_{m+1-b,m+1-a}$ we see that we may as usual assume $a + b \geqslant m + 1$. Below we will show that (7.12.1) yields a minimal presentation of C_{ab} provided we drop the projective Q. Furthermore we give an explicit matrix representation for ρ.

In characteristic zero our presentation can be block diagonalized yielding a decomposition of $\operatorname{Hom}_R(M_a, M_b)$ into certain maximal Cohen-Macaulay modules of lower rank.

9.1. *A star product.* We first recall a well-known formula of Gerstenhaber and Schack. Assume that $\psi_1, \dots, \psi_n, \theta_1, \dots, \theta_n$ are commuting nilpotent derivations on a commutative algebra A containing $\mathbb{Q}$. Then, denoting by $\mathfrak{m}\colon A \otimes A \longrightarrow A$ the multiplication in A, there is an associated associative product

$$u * v = \mathfrak{m}(e^{\psi_1\otimes\theta_1 + \cdots \psi_n\otimes\theta_n}(u \otimes v))$$

on A. It is easy to see that this formula generalizes to the graded case.

Applying this formula with $A = \bigwedge_S(\mathcal{F}^\vee \oplus \mathcal{G})$ and

$$\psi_i = \frac{\partial}{\partial g_i}\,, \qquad\qquad \theta_i = -\sum_{j=1}^{m} x_{ji}\frac{\partial}{\partial\lambda_j}$$

for $i = 1, \dots, n$, yields a multiplication on A via

$$u * v = \mathfrak{m}(e^{-\Delta}(u \otimes v)) \tag{9.1.1}$$

where

$$\Delta = \sum_{\substack{i=1,\dots,n\\ j=1,\dots,m}} x_{ji}\frac{\partial}{\partial g_i} \otimes \frac{\partial}{\partial\lambda_j}\,.$$

Lemma 9.2. *The star product on $A = \bigwedge_S(\mathcal{F}^\vee \oplus \mathcal{G})$ gives A the structure of a quadratic S-algebra generated by the symbols $\lambda_1, \dots, \lambda_m, g_1, \dots, g_n$ subject to the relations*

$$\begin{aligned}
&\lambda_k * \lambda_l = \lambda_k\lambda_l = -\lambda_l\lambda_k = -\lambda_l * \lambda_k, && \lambda_k * \lambda_k = \lambda_k^2 = 0,\\
&g_k * g_l = g_k g_l = -g_l g_k = -g_l * g_k, && g_k * g_k = g_k^2 = 0,\\
&g_k * \lambda_l = g_k\lambda_l + x_{kl} = -\lambda_l * g_k + x_{kl}\,.
\end{aligned}$$

In other terms, $(A, *)$ is isomorphic to the Clifford algebra C on $F^\vee$ and G.

We quickly show that in this particular case (9.1.1) is defined over $\mathbb{Z}$ and thus is true in arbitrary characteristic. To this end we have to compute Δ^t. We find

$$\begin{aligned}\Delta^t &= \sum x_{j_1 i_i} \cdots x_{j_t i_t} \frac{\partial}{\partial g_{i_t}} \cdots \frac{\partial}{\partial g_{i_1}} \otimes \frac{\partial}{\partial \lambda_{j_1}} \cdots \frac{\partial}{\partial \lambda_{j_t}} \\ &= t! \sum_{j_1 < \cdots < j_t} x_{j_1 i_i} \cdots x_{j_t i_t} \frac{\partial}{\partial g_{i_t}} \cdots \frac{\partial}{\partial g_{i_1}} \otimes \frac{\partial}{\partial \lambda_{j_1}} \cdots \frac{\partial}{\partial \lambda_{j_t}} \\ &= t! \sum_{\substack{i_1 < \cdots < i_t \\ j_1 < \cdots < j_t}} [i_1 \cdots i_t \mid j_1 \cdots j_t] \frac{\partial}{\partial g_{i_t}} \cdots \frac{\partial}{\partial g_{i_1}} \otimes \frac{\partial}{\partial \lambda_{j_1}} \cdots \frac{\partial}{\partial \lambda_{j_t}}\end{aligned}$$

where the peculiar arrangement of indices is to eliminate some signs and where $[i_1 \cdots i_t \mid j_1 \cdots j_t]$ is the (unsigned) determinant of the $(t \times t)$-submatrix of X consisting of the rows indexed $i_1, \ldots, i_t$ and columns indexed $j_1, \ldots, j_t$.

It follows that if we set

$$\Delta^{(t)} = \frac{\Delta^t}{t!} = \sum_{\substack{i_1 < \cdots < i_t \\ j_1 < \cdots < j_t}} [i_t \cdots i_1 \mid j_t \cdots j_1] \frac{\partial}{\partial g_{i_t}} \cdots \frac{\partial}{\partial g_{i_1}} \otimes \frac{\partial}{\partial \lambda_{j_1}} \cdots \frac{\partial}{\partial \lambda_{j_t}} \,,$$

then the star product on $A = \bigwedge_S(\mathcal{F}^\vee \oplus \mathcal{G})$ is given by

$$\mathfrak{m} \circ \left(1 - \Delta + \Delta^{(2)} - \cdots\right) \,.$$

Return now to the free S-presentation of C_{ab} given by Lemma 7.12. We have the following simplification of this presentation

Proposition 9.3. *If $a + b \geqslant m + 1$ then C_{ab} has a minimal S-free presentation of the form*

$$P_1 \xrightarrow{\rho} P_0 \longrightarrow C_{ab} \longrightarrow 0 \,, \tag{9.3.1}$$

where

- $P_0 = \bigoplus_{\max\{a,b\} \leqslant k \leqslant m} \bigwedge_S^{k-a} \mathcal{G} \otimes \bigwedge_S^{k-b} \mathcal{F}^\vee$.
- $P_1 = \bigoplus_{0 \geqslant l \geqslant \max\{a-m, b-m\}} \bigwedge_S^{b-l} \mathcal{G} \otimes \bigwedge_S^{a-l} \mathcal{F}^\vee$
- $\rho_{lk} = \begin{cases} (\Delta^{(a+b-k-l)})_{lk} & \textit{if } a+b-k-l \geqslant 0, \textit{ and} \\ 0 & \textit{otherwise.} \end{cases}$.

Proof. Our starting point is the free presentation of C_{ab} given in (7.12.1). It takes the form

$$\bigoplus_{1 > l \geqslant \max\{a-m, b-n\}} \bigwedge_S^{b-l} \mathcal{G} \otimes \bigwedge_S^{a-l} \mathcal{F}^\vee \longrightarrow \bigoplus_{\max\{a,b\} \leqslant k \leqslant m} \bigwedge_S^{k-b} \mathcal{F}^\vee \otimes \bigwedge_S^{k-a} \mathcal{G} \tag{9.3.2}$$

where ρ is obtained by expanding paths that go first to the left and then to the right in terms of paths that do the opposite.

Now we borrow some ingredients from the proof of Theorem 7.15. Writing (9.3.2) in the form

$$Q \oplus P_1 \xrightarrow{\rho} P_0$$

as in (7.12.1) we deduce from the fact that α_1 is shown to be invertible in (7.15.8) that ρ and $(\rho \mid_{P_1})\colon P_1 \longrightarrow P_0$ represent the same S-module. This shows that C_{ab} has a presentation as in (9.3.1). Furthermore the resulting matrix entry

$$\bigwedge_S^{b-l} \mathcal{G} \otimes \bigwedge_S^{a-l} \mathcal{F}^\vee \longrightarrow \bigwedge_S^{k-b} \mathcal{F}^\vee \otimes \bigwedge_S^{k-a} \mathcal{G}$$

can be deduced by working in $(\bigwedge_S(\mathcal{F}^\vee \oplus \mathcal{G}), *)$. We find that it is the composition

$$(9.3.3)\qquad \textstyle\bigwedge_S^{b-l}\mathcal{G}\otimes\bigwedge_S^{a-l}\mathcal{F}^\vee \xrightarrow{(-1)^{a+b-k-l}\Delta^{(a+b-k-l)}} \bigwedge_S^{k-a}\mathcal{G}\otimes\bigwedge_S^{k-b}\mathcal{F}^\vee \xrightarrow{(-1)^{(k-b)(k-a)}} \bigwedge_S^{k-b}\mathcal{F}^\vee\otimes\bigwedge_S^{k-a}\mathcal{G}\,.$$

The presentation given in the statement of the proposition is deduced from this by pre- and postcomposing with invertible diagonal matrices (with diagonal entries in $\{\pm 1\}$).

The presentation is minimal if and only if $a+b-k-l\geqslant 1$ for all allowable k,l. It is enough to test this for k,l maximal, i.e. $k=m$, $l=0$. Then $a+b-k-l=a+b-m$ which is positive if and only if $a+b\geqslant m+1$. $\square$

Example 9.4. Assume that $m=n=5$, $a=b=4$. Then

$$P_0=\bigoplus_{4\leqslant k\leqslant 5}\textstyle\bigwedge_S^{k-4}\mathcal{G}\otimes\bigwedge_S^{k-4}\mathcal{F}^\vee = K\oplus\mathcal{G}\otimes\mathcal{F}^\vee$$

$$P_1=\bigoplus_{0\geqslant l\geqslant -1}\textstyle\bigwedge_S^{4-l}\mathcal{G}\otimes\bigwedge_S^{4-l}\mathcal{F}^\vee=\bigwedge_S^5\mathcal{G}\otimes\bigwedge_S^5\mathcal{F}^\vee\oplus\bigwedge_S^4\mathcal{G}\otimes\bigwedge_S^4\mathcal{F}^\vee$$

and the matrix form of the presentation is

$$\begin{pmatrix}\Delta^{(5)} & \Delta^{(4)}\\ \Delta^{(4)} & \Delta^{(3)}\end{pmatrix}$$

It will be clear to the reader that over $\mathbb{Q}$ this presentation can be diagonalized further. We will say more on this below.

9.5. *Characteristic zero.* In this section we assume $\operatorname{char} K=0$. For $\alpha,\beta\geqslant 0$ with $\alpha+\beta<m$, define $C^{\alpha\beta}$ to be the cokernel of

$$\Delta^{(m-\alpha-\beta)}\colon \textstyle\bigwedge_S^{m-\beta}\mathcal{G}\otimes\bigwedge_S^{m-\alpha}\mathcal{F}^\vee\longrightarrow\bigwedge_S^{\alpha}\mathcal{G}\otimes\bigwedge_S^{\beta}\mathcal{F}^\vee\,.$$

Proposition 9.6. *Assume* $\operatorname{char} K=0$ *and that* $a+b\geqslant m+1$. *Then*

(1) *The* $C^{\alpha\beta}$ *are maximal Cohen-Macaulay* R*-modules.*
(2) *We have a decomposition*

$$C_{ab}=\bigoplus_{\max\{a,b\}\leqslant p\leqslant m} C^{p-a,p-b}\,.$$

Proof. According to Proposition 9.3, the map ρ written as a matrix has the form

$$(9.6.1)\qquad \rho=\begin{pmatrix}\Delta^{(r)} & \Delta^{(r-1)} & \cdots & \\ \Delta^{(r-1)} & \cdots & & \\ \vdots & & & \vdots\\ & & \cdots & \Delta^{(s)}\end{pmatrix}=\begin{pmatrix}\frac{\Delta^r}{r!} & \frac{\Delta^{r-1}}{(r-1)!} & \cdots & \\ \frac{\Delta^{r-1}}{(r-1)!} & \cdots & & \\ \vdots & & & \vdots\\ & & \cdots & \frac{\Delta^s}{s!}\end{pmatrix}.$$

Here $\Delta^{(r)}$ represents the map from $\bigwedge_S^{b-l}\mathcal{G}\otimes\bigwedge_S^{a-l}\mathcal{F}^\vee$ to $\bigwedge_S^{k-a}\mathcal{G}\otimes\bigwedge_S^{k-b}\mathcal{F}^\vee$ for $k=\max\{a,b\}$ and $l=\max\{a,b\}-m$. Thus $r=a+b-2\max\{a,b\}+m=-|a-b|+m$.

Similarly $\Delta^{(s)}$ represents the map $\bigwedge_S^{b-l}\mathcal{G}\otimes\bigwedge_S^{a-l}\mathcal{F}^\vee$ to $\bigwedge_S^{k-a}\mathcal{G}\otimes\bigwedge_S^{k-b}\mathcal{F}^\vee$ for $k=m$, $l=0$. Thus $s=a+b-m$.

It order to manipulate (9.6.1) we write it *formally* as

$$\rho = \begin{pmatrix} \Delta^{r/2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta^{s/2} \end{pmatrix} \begin{pmatrix} \frac{1}{r!} & \frac{1}{(r-1)!} & \cdots & \\ \frac{1}{(r-1)!} & \ddots & & \\ \vdots & & & \vdots \\ & & \cdots & \frac{1}{s!} \end{pmatrix} \begin{pmatrix} \Delta^{r/2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta^{s/2} \end{pmatrix}.$$

Let A be the middle scalar matrix. According to Lemma 9.9 below we have $A = PDP^t$ where D is a non-singular diagonal matrix and P is upper triangular with 1's on the diagonal, both with rational entries.

Let $\widetilde{P}$ be obtained from P by replacing P_{ij} by $\Delta^{j-i}P_{ij}$. Then after a bit of manipulation we obtain the following (non-formal) expression for ρ.

$$\rho = \widetilde{P} \begin{pmatrix} D_{rr}\frac{\Delta^r}{r!} & 0 & \cdots & \\ 0 & \ddots & & \\ \vdots & & & \vdots \\ & & \cdots & D_{ss}\frac{\Delta^s}{s!} \end{pmatrix} \widetilde{P}^t$$

As $\widetilde{P}$ is invertible, this shows that C_{ab} indeed has a decomposition as indicated in the statement of the proposition. If follows that $C^{\alpha\beta}$ is a maximal Cohen-Macaulay R-module if $C^{\alpha\beta}$ occurs as a summand among one of the C^{ab}. Given $\alpha, \beta \geqslant 0$, with $\alpha + \beta < m$, we put $p = m$ so that $a = m - \alpha$, $b = m - \beta$. Then $a + b = 2m - (\alpha + \beta) \geqslant m + 1$, as required. □

Example 9.7. The following matrix gives the decomposition of C_{ab} for $m = 3$ (and n arbitrary).

$$\begin{pmatrix} C^{00} & C^{10} & C^{20} \\ C^{01} & C^{00} \oplus C^{11} & C^{10} \\ C^{02} & C^{01} & C^{00} \end{pmatrix}$$

The cases $a + b \geqslant m + 1 = 4$ are covered by the proposition. For the other cases we perform the involution $(a, b) \mapsto (m + 1 - b, m + 1 - a) = (4 - b, 4 - a)$.

The following lemma is used in the next lemma, which was used in the above proof.

Lemma 9.8. *Let $u \geqslant 2t$ and let A be the $(t \times t)$-matrix over $\mathbb{Q}$*

$$A_{ij} = \frac{1}{(u - i - j)!}$$

with $1 \leqslant i, j \leqslant t$. Then $\det A \neq 0$.

Proof. Put $B = (u - 2)!A$. Then B is equal to

$$\begin{pmatrix} 1 & x & x(x-1) & \cdots & x(x-1)\cdots(x-t+2) \\ x & x(x-1) & x(x-1)(x-2) & \cdots & x(x-1)\cdots(x-t+1) \\ x(x-1) & x(x-1)(x-2) & x(x-1)(x-2)(x-3) & \cdots & x(x-1)\cdots(x-t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

with $x = u - 2$. Then

$$\begin{aligned} \det B &= x \cdot x(x-1) \cdot x(x-1)(x-2) \cdots x(x-1)\cdots(x-t+2) \det C \\ &= x^{t-1}(x-1)^{t-2} \cdots (x - t + 2) \det C \end{aligned}$$

with C being equal to

$$\begin{pmatrix} 1 & x & x(x-1) & \cdots & x(x-1)\cdots(x-t+2) \\ 1 & x-1 & (x-1)(x-2) & \cdots & (x-1)\cdots(x-t+1) \\ 1 & x-2 & (x-2)(x-3) & \cdots & (x-2)\cdots(x-t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$

If we put $x_i = x - i$ then C can be written as

$$\begin{pmatrix} 1 & x_0 & x_0(x_0-1) & \cdots & x_0(x_0-1)\cdots(x_0-t+2) \\ 1 & x_1 & x_1(x_1-1) & \cdots & x_1\cdots(x_1-t+2) \\ 1 & x_2 & x_2(x_2-1) & \cdots & x_2\cdots(x_2-t+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

which using column operations can be turned into a Vandermonde determinant. Hence

$$\det C = \prod_{0 \leqslant i < j \leqslant t-1} (x_j - x_i) = \prod_{0 \leqslant i < j \leqslant t-1} (i-j) \neq 0\,.$$

□

Lemma 9.9. *Let A be as in the previous lemma. Then $A = PDP^t$ with D diagonal and P upper triangular with 1's on the diagonal.*

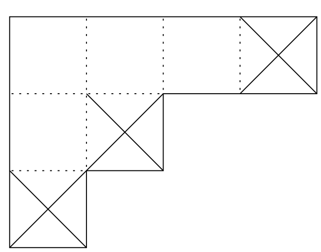
Proof. We view A, being a symmetric matrix, as a quadratic form. Diagonalizing it in the usual way, starting with the last variable, we see that we need $\det(A_{ij})_{p \leqslant i,j \leqslant t} \neq 0$ for $p = 1, \dots, t$. This follows from the previous lemma. □

10. Minimal Resolutions of the Simples in Characteristic Zero

In this final section we require K to be a field of characteristic zero. Other than that, we keep the established notation. Our aim in this section is to compute the Ext-groups among the graded simple modules over the non-commutative desingularization E, and so obtain the shapes of their minimal graded free resolutions.

10.1. *The main result.* We follow the notation of Weyman [24] for Schur modules L_α corresponding to partitions $\alpha = (\alpha_1, \dots, \alpha_q)$. Let $\Gamma(m,n)$ be the set of Young diagrams (identified, as usual, with partitions) having at most m rows and n columns. The conjugate partition α' is obtained by reflection across the line $y = -x$.

A *convex square* of a diagram $\alpha \in \Gamma(m,n)$ is a square with coordinates (r, α_r) such that $\alpha_{r+1} < \alpha_r$. For a convex square (r,c) in α, let $R_r(\alpha)$ be the partition obtained from α by dropping the r^{th} row. Similarly, $C_c(\alpha)$ is obtained by dropping α's c^{th} column. For example, we have indicated below the convex squares for the partition (421).



We obtain three corresponding pairs of partitions $(C_c(\alpha), R_r(\alpha))$, namely $((321),(21))$, $((311),(41))$, and $((31),(42))$.

For $a = 1,\dots,m$ let $P_a = \operatorname{Hom}_R(M_a, M)$ be the corresponding graded projective left E-module and let S_a be the associated graded simple module. We have:

Theorem 10.2. *Assume* $\operatorname{char} K = 0$. *For simple right E-modules S_a and S_b, we have*

$$\operatorname{Ext}^t_E(S_b, S_a) \cong \bigoplus L_{C_c(\alpha)}F \otimes L_{R_r(\alpha)'}G^\vee\,,$$

where the direct sum is taken over all partitions $\alpha \in \Gamma(m,n)$ such that $|\alpha| = t+1$, and over all convex squares (r,c) in α such that $-a+b = -r+c$.

The proof of this theorem will occupy the remainder of the section.

Example 10.3. We can evaluate the sum above for small values of t, obtaining the first few terms of the resolution of S_a:

$$P_a \longleftarrow \begin{matrix} P_{a-1}(-1)\otimes F^\vee \\ \oplus \\ P_{a+1}(-1)\otimes G \end{matrix} \longleftarrow \begin{matrix} P_{a-2}(-2)\otimes \mathbb{S}^2F^\vee \\ \oplus \\ P_{a-1}(-3)\otimes \bigwedge^2F^\vee\otimes G \\ \oplus \\ P_{a+1}(-3)\otimes F^\vee\otimes \bigwedge^2 G \\ \oplus \\ P_{a+2}(-2)\otimes \mathbb{S}^2G \end{matrix} \longleftarrow \begin{matrix} P_{a-3}(-3)\otimes \mathbb{S}^3F^\vee \\ \oplus \\ P_{a-2}(-4)\otimes L_{21}F^\vee\otimes G \\ \oplus \\ P_{a-1}(-5)\otimes \bigwedge^3F^\vee\otimes \mathbb{S}^2G \\ \oplus \\ P_{a}(-4)\otimes \bigwedge^2F^\vee\otimes \bigwedge^2G \\ \oplus \\ P_{a+1}(-5)\otimes \mathbb{S}^2F^\vee\otimes \bigwedge^3G \\ \oplus \\ P_{a+2}(-4)\otimes F^\vee\otimes L_{21}G \\ \oplus \\ P_{a+3}(-3)\otimes \mathbb{S}^3G \end{matrix}$$

where we understand $P_i = 0$ if $i \notin [1,m]$. From this resolution we can read off the generators and relations of $C \cong E$. Of course, the result is consistent with Remark 7.6. The interpretation of the higher terms in the resolution remains open.

10.4. *Translation into geometry.* As a matter of notational convenience in this section, we dualize and work with the *twisted tangent bundle* $\mathcal{Q} := \mathcal{U}^\vee = (\Omega^1(1))^\vee$ on $\mathbb{P} = \mathbb{P}(F^\vee)$ defined by exactness of the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow F^\vee\otimes\mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{Q} \longrightarrow 0\,.$$

With the same argument as in Theorem 6.4 it follows that $\mathcal{M}' = p'^*(\bigwedge\mathcal{Q})$ is also a tilting bundle on $\mathcal{Z}$. In particular, we have the exact equivalence of categories

$$\mathbf{R}\operatorname{Hom}_{\mathcal{O}_{\mathcal{Z}}}(p'^*(\textstyle\bigwedge\mathcal{Q}), -) \colon \mathcal{D}^b(\operatorname{coh}(\mathcal{Z})) \longrightarrow \mathcal{D}(E)$$

since $\operatorname{End}_{\mathcal{O}_{\mathcal{Z}}}(p'^*(\bigwedge\mathcal{Q})) \cong \operatorname{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T})^{\mathrm{op}} = E^{\mathrm{op}}$. This equivalence sends each $p'^*(\bigwedge^a\mathcal{Q})$, $a = 0,\dots,m-1$, to the graded projective left E-module P_{a+1}.

Lemma 10.5. *Let $u\colon \mathbb{P} \longrightarrow \mathcal{Z}$ be the zero section of the vector bundle $p'\colon \mathcal{Z} \longrightarrow \mathbb{P}$ (see §5.2). The object in $\mathcal{D}^b(\operatorname{coh}(\mathcal{Z}))$ corresponding to the simple module S_{a+1} is $u_*\mathcal{O}_{\mathbb{P}}(-a)[a]$.*

Proof. We must show that $\operatorname{Ext}^t_{\mathcal{O}_{\mathcal{Z}}}(p'^*\bigwedge^b\mathcal{Q}, u_*\mathcal{O}_{\mathbb{P}}(-a)[a])$ is one-dimensional if $t=0$ and $a=b$, and vanishes otherwise. By adjunction it suffices to prove

$$\operatorname{Ext}^t_{\mathcal{O}_{\mathbb{P}}}\left(\textstyle\bigwedge^b\mathcal{Q}, \mathcal{O}_{\mathbb{P}}(-a)\right) = H^t(\mathbb{P}, \Omega^b(b-a)) = \begin{cases} K & \text{if } t=a=b, \text{and} \\ 0 & \text{otherwise.}\end{cases}$$

Computing
$$\Omega^b(b-a) = \mathcal{M}^m_{b+1}(-a-1)\otimes |F|^\vee ,$$
we finish the proof by invoking Theorem 3.9. □

Hence in order to prove Theorem 10.2 it is sufficient to compute
$$\operatorname{Ext}^t_{\mathcal{O}_\mathcal{Z}}\left(u_*\mathcal{O}_\mathbb{P}(-b)[b], u_*\mathcal{O}_\mathbb{P}(-a)[a]\right) = \operatorname{Ext}^{t-b+a}_{\mathcal{O}_\mathcal{Z}}\left(u_*\mathcal{O}_\mathbb{P}(-b), u_*\mathcal{O}_\mathbb{P}(-a)\right) .$$
To this end we prove something more general.

Proposition 10.6. *Let $\mathcal{U},\mathcal{V}$ be objects in $\mathcal{D}^b(\mathrm{coh}(\mathbb{P}))$. Then*
$$\operatorname{Ext}^t_{\mathcal{O}_\mathcal{Z}}\left(u_*\mathcal{U}, u_*\mathcal{V}\right) = \bigoplus_s \operatorname{Ext}^{t-s}_{\mathcal{O}_\mathbb{P}}\left(\bigwedge^s(\mathcal{Q}\otimes G)\otimes_\mathbb{P}\mathcal{U},\mathcal{V}\right) .$$

Proof. We may assume that $\mathcal{U}$ is a bounded complex of locally free $\mathcal{O}_\mathbb{P}$-modules. The locally free resolution of $u_*\mathcal{U}$ as $\mathcal{O}_\mathcal{Z}$-module is then given by
$$\cdots\longrightarrow \bigwedge^2(\mathcal{Q}\otimes G)\otimes_{\mathcal{O}_\mathbb{P}}\mathcal{U}\otimes_{\mathcal{O}_\mathbb{P}}\mathcal{O}_\mathcal{Z}\longrightarrow \mathcal{Q}\otimes G\otimes_{\mathcal{O}_\mathbb{P}}\mathcal{U}\otimes_{\mathcal{O}_\mathbb{P}}\mathcal{O}_\mathcal{Z}\longrightarrow \mathcal{U}\otimes_{\mathcal{O}_\mathbb{P}}\mathcal{O}_\mathcal{Z}\longrightarrow 0 .$$
It follows that $\mathbf{R}\mathscr{H}om_{\mathcal{O}_\mathcal{Z}}(\mathcal{U},\mathcal{V})$ is equal in $\mathcal{D}^b(\mathrm{coh}(\mathbb{P}))$ to the complex
$$0\longrightarrow \mathscr{H}om_{\mathcal{O}_\mathbb{P}}(\mathcal{U},\mathcal{V})\longrightarrow \mathscr{H}om_{\mathcal{O}_\mathbb{P}}(\mathcal{Q}\otimes G\otimes_{\mathcal{O}_\mathbb{P}}\mathcal{U},\mathcal{V})\longrightarrow \mathscr{H}om_{\mathcal{O}_\mathbb{P}}(\bigwedge^2(\mathcal{Q}\otimes G)\otimes_{\mathcal{O}_\mathbb{P}}\mathcal{U},\mathcal{V})\longrightarrow\cdots .$$
We note however that the center of $\mathrm{GL}(G)$ acts with different weights on the terms of this complex. It follows that the maps are necessarily all zero, whence
$$\mathbf{R}\mathscr{H}om_{\mathcal{O}_\mathcal{Z}}(\mathcal{U},\mathcal{V}) = \bigoplus_s \mathscr{H}om_{\mathcal{O}_\mathbb{P}}\left(\bigwedge^s(\mathcal{Q}\otimes G)\otimes_{\mathcal{O}_\mathbb{P}}\mathcal{U},\mathcal{V}\right)[-s] .$$
This implies the form claimed. □

Proof of Theorem 10.2. From Lemma 10.5 and Prop. 10.6 we obtain
$$\begin{aligned}\operatorname{Ext}^t_E(S_b,S_a) &= \operatorname{Ext}^{t-b+a}_{\mathcal{O}_\mathcal{Z}}\left(u_*\mathcal{O}_\mathbb{P}(-b+1)[b-1], u_*\mathcal{O}_\mathbb{P}(-a+1)[a-1]\right)\\ &= \bigoplus_s H^{t-b+a-s}\left(\mathbb{P},\bigwedge^s(\mathcal{Q}\otimes G)^\vee(b-a)\right) .\end{aligned}$$
Expanding $\bigwedge^s(\mathcal{Q}\otimes G)$ according to the Cauchy formula (this is the first time we use $\operatorname{char} K=0$)
$$\bigwedge^s(\mathcal{Q}\otimes G) = \bigoplus_{|\alpha|=s} L_\alpha\mathcal{Q}\otimes L_{\alpha'}G$$
we find
$$\operatorname{Ext}^t_E(S_b,S_a) = \bigoplus_\alpha H^{t-b+a-|\alpha|}(\mathbb{P},(L_\alpha\mathcal{Q})^\vee(b-a))\otimes L_{\alpha'}G^\vee .$$
To continue, we apply Serre duality:
$$H^{t-b+a-|\alpha|}(\mathbb{P},(L_\alpha\mathcal{Q})^\vee(b-a)) = H^{m-1-t+b-a+|\alpha|}(\mathbb{P},(L_\alpha\mathcal{Q})(a-b-m))^\vee\otimes|F^\vee| .$$
Using a straightforward application of Bott's theorem (see the discussion after the current proof) the direct sum can now be written as

$$\operatorname{Ext}^t_E(S_b,S_a) = \bigoplus_{\substack{\alpha<_{m-a+b}\beta\\ l(\beta-\alpha)=m-t-a+b+|\alpha|}} L_\beta F\otimes L_{\beta'}G^\vee\otimes|F^\vee| \tag{10.6.1}$$

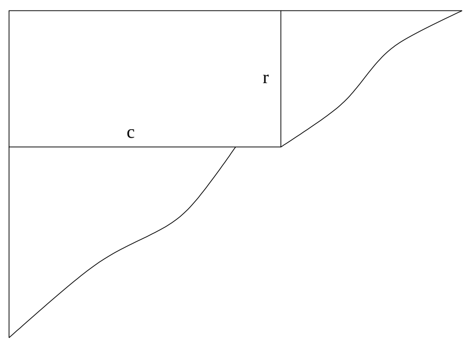
$$= \bigoplus_{\substack{\alpha<_{m-a+b}\beta\\ c(\beta-\alpha)=t-|\alpha|+1}} L_\beta F\otimes L_{\beta'}G^\vee\otimes|F^\vee| \tag{10.6.2}$$

where the notation $\alpha <_s \beta$ means that $\beta - \alpha$ is a *rim hook* (or *border strip*) of length s ending at the m^{th} row. Recall that a rim hook is a connected skew tableau not containing any (2×2)-squares. We write $l(\beta - \alpha)$ for the number of rows in $\beta - \alpha$, and $c(\beta - \alpha)$ for the number of columns.

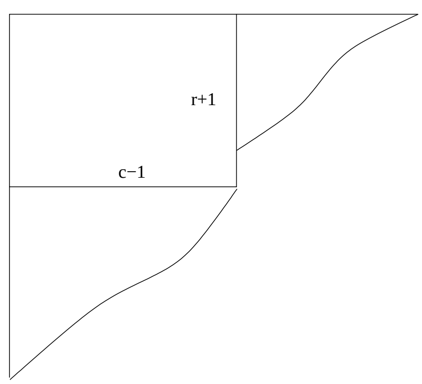
The formula (10.6.1) can be expressed symmetrically as

$$\operatorname{Ext}^t_E(S_b, S_a) = \bigoplus_{\substack{\mu \cong_{c,r} \nu \\ t - |\nu| = c-1 \\ -a+b = -r+c-1}} L_\mu F \otimes L_{\nu'} G^\vee \,. \tag{10.6.3}$$

In this sum μ runs over partitions with at most $m-1$ rows and m columns, while ν runs over those with at most m rows and $m-1$ columns. The notation $\mu \cong_{c,r} \nu$ indicates that ν contains an embedded $r \times c$ rectangle as shown



with $r \geqslant 0$, $c > 0$, and μ is obtained by replacing the rectangle by an $(r+1) \times (c-1)$ rectangle.



It is now easy to obtain the statement of Theorem 10.2 from (10.6.3), completing the proof. □

10.7. In the previous proof we have used Bott's theorem for which we provide a brief reminder to the reader. Let G be a reductive group and let $T \subset B \subset P \subset G$ be respectively a maximal torus T, a Borel subgroup B and a parabolic subgroup P. For a dominant weight $\theta \in X(T)$ let L^G_θ be the corresponding simple G-representation.

Taking fibers in $[P] \in G/P$ provides an equivalence between rational P-representations and G-equivariant quasi-coherent sheaves on G/P. Denote this equivalence by $\widetilde{?}$. Let $H = P/\operatorname{rad} P$ be the reductive part of P and let L^H_χ be the simple

H-representation associated to a H-dominant weight $\chi \in X(T)$. We view L^H_χ as a P-representation.

Bott's theorem computes the cohomology of $\widetilde{L}^H_\chi$ as follows

(10.7.1)

$$H^i(G/P, \widetilde{L}^H_\chi) = \begin{cases} L^G_\theta & \text{if there exists a (necessarily unique) } w \in W \text{ such} \\ & \text{that } \theta = w \cdot \xi \text{ is } G\text{-dominant and } l(w) = i \\ 0 & \text{otherwise.} \end{cases}$$

where W is the Weyl group of G and where $w \cdot \xi = w(\xi + \rho) - \rho$ is the twisted Weyl group action, with ρ as usual being half the sum of the positive roots.

Now in the setting of this paper choose an identification $F^\vee = K^m$ and let $G = \mathrm{GL}_m(K)$. Then $\mathbb{P}(F^\vee) = G/P$ where P is the stabilizer of the point $p = (0, \ldots, 0, 1)$. Let $T = \{\mathrm{diag}(t_1, \ldots, t_m)\} \subset G$ be the diagonal torus. We view $t_1, \ldots, t_m$ as characters of T.

The roots of G are $t_i t_j^{-1}$, $i \neq j$, with the positive roots being those for which $i > j$ (in this setting the negative roots are the non-zero weights of $\mathrm{Lie}(B)$). The G-dominant weights are of the form $t_1^{\alpha_1} \cdots t_m^{\alpha_m}$ with $\alpha_1 \geqslant \ldots \geqslant \alpha_m$. Thus the dominant weights α are partitions with at most m rows and one has $L^G_\alpha = L_\alpha F^\vee$. The (twisted) action of the Weyl group is generated by the reflections

$$s_i \colon t_i^{\alpha_i} t_{i+1}^{\alpha_{i+1}} \mapsto t_i^{\alpha_{i+1}-1} t_{i+1}^{\alpha_i+1}\,. \tag{10.7.2}$$

The G-equivariant exact sequence

$$0 \longrightarrow \mathcal{O}_\mathbb{P}(-1) \longrightarrow F^\vee \otimes \mathcal{O}_\mathbb{P} \longrightarrow \mathcal{Q} \longrightarrow 0$$

yields a P-equivariant exact sequence

$$0 \longrightarrow \mathcal{O}_\mathbb{P}(-1)_p \longrightarrow F^\vee \longrightarrow \mathcal{Q}_p \longrightarrow 0$$

with $\dim \mathcal{O}_\mathbb{P}(-1)_p = 1$, $\dim \mathcal{Q}_p = m - 1$. Such an exact sequence is unique and must be isomorphic to

$$0 \longrightarrow K \longrightarrow K^m \longrightarrow K^{m-1} \longrightarrow 0$$

where the first non-trivial map is the injection into the last factor and the second non-trivial map is the projection onto the first $m-1$ factors. This means that $\mathcal{O}_\mathbb{P}(-1) = \widetilde{L}^H_{t_m}$, $\mathcal{Q} = \widetilde{L}^H_{t_1}$ where $H = \mathrm{GL}_{m-1}(K) \times \mathrm{GL}_1(K)$. Looking at the stalk in p we also compute that for a partition α with at most $m-1$ rows we have $L_\alpha \mathcal{Q}(-s) = \widetilde{L}^H_{t_1^{\alpha_1} \cdots t_{m-1}^{\alpha_{m-1}} t_m^s}$. Hence to compute the cohomology of $L_\alpha \mathcal{Q}(-s)$ using (10.7.1) we must try to flatten the factor t_m^s in the weight $t_1^{\alpha_1} \cdots t_{m-1}^{\alpha_{m-1}} t_m^s$ using the twisted Weyl group action (10.7.2). We see that this is only possible if there is a partition β with m rows such that $\beta - \alpha$ is a rim hook with s boxes and the number of reflections we need in that case is one less than the number of rows in $\beta - \alpha$. This completes the derivation of (10.6.1).

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