Factoring the Adjoint and MCM Modules over the Generic Determinant

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$k =$ Noetherian base ring
$X = (x_{ij})$, the generic $n \times n$ matrix over k

Recall: adj(X) is the "classical adjoint" of X

$$X \ \text{adj}(X) = \det(X) \cdot I_n = \text{adj}(X) \ X$$

Question (G. Bergman): Can this factorization be refined?

$$\det(X) \cdot I_n \stackrel{?}{=} \underbrace{(Y_1 \cdots Y_r)}_{X} \underbrace{(Z_1 \cdots Z_s)}_{\mathrm{adj}(X)}$$

Since $\det(X)$ is irreducible, X cannot be factored, so the real question is

Can $\mathrm{adj}(X)$ be written as

$$\mathrm{adj}(X) = YZ$$

for noninvertible square matrices

Y and Z?

Theorem (Bergman): Assume k is an algebraically closed field, char 0.

(a) For n odd, adj(X) can't be factored.

(b) If n is even, any factorization

$$\text{adj}(X) = YZ$$

has $\det(Y) = \det(X)$ or $\det(Z) = \det(X)$.

Proof uses the "Hedgehog Theorem" from topology, and generalizations by De Concini – Reichstein.

Why do we care?

Recall: A *matrix factorization*

$$AB = f \cdot I_n = BA\,,$$

of $f \in k[x_{ij}]$, corresponds to a *maximal Cohen–Macaulay module* over $R = k[x_{ij}]/(f)$.

(M is MCM if $\operatorname{depth} M = \dim R$.)

Correspondence:

$$(A, B) \leftrightarrow \operatorname{cok}(A)$$

So in particular

$$L := \mathsf{cok}\, X$$

and

$$M := \mathsf{cok}(\mathsf{adj}(X))$$

are MCM modules over

$$R = k[x_{ij}]/(\mathsf{det}(X))$$

of ranks 1 and $n-1$, respectively. They're syzygies of each other, both indecomposable and nonfree.

When $n = 2$:

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

and

$$\text{adj}(X) = \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}$$

So

$$M = \text{cok}(\text{adj}(X)) \cong \text{cok}(X^T) \cong L^*.$$

(dual over R)

When $n = 2$, R has *finite CM type*: the only indecomposable MCMs are R, L and L^*.

When $n > 2$, R is not an isolated singularity, so has infinitely many MCMs [Auslander].

However, there are still only 3 rank-one MCMs: R, L, L^*.

Question: Are there only finitely many MCMs of each rank? Or maybe nice parametrized families?

How do the factorizations come in?

A factorization $\mathrm{adj}(X) = YZ$:

$$\begin{array}{ccc} S^n & \xrightarrow{\mathrm{adj}(X)} & S^n \\ & {\scriptstyle Z}\searrow \quad \nearrow{\scriptstyle Y} & \\ & S^n & \end{array}$$

gives a short exact sequence

$$0 \longrightarrow \mathrm{cok}(Z) \longrightarrow M \longrightarrow \mathrm{cok}(Y) \longrightarrow 0$$

with $\mathrm{cok}(Z)$ and $\mathrm{cok}(Y)$ MCM R-modules (by the Ker-Coker Lemma).

Bergman's question becomes:

Does the cokernel
of the adjoint matrix
appear as an extension of
two MCM R-modules?

Prop. When $n = 3$ and k is any normal domain, No.

Proof. cok(adj(X)) has rank $n-1 = 2$, so the ends would have rank 1. Now compute in the divisor class group. $\square$

By Bergman's Theorem, we should expect $\mathrm{adj}(X) = YZ$ only when either $\det(Y) = \det(X)$ or $\det(Z) = \det(X)$.

Equivalently, either $\mathrm{cok}(Y)$ or $\mathrm{cok}(Z)$ has rank 1.

But the rank-1 MCMs are known: just R, L, and L^*.

Use the fact that $M = \mathrm{syz}_1(L)$, and a pushout diagram, to get an extension

$$0 \longrightarrow \mathrm{cok}(Y) \longrightarrow Q \longrightarrow L \longrightarrow 0$$

with $\mathrm{cok}(Y) \cong$ either L or L^*,
and $\mathrm{rank}(Q) = 2$.

Assume k is a field.

Theorem: $\mathrm{Ext}^1_R(L, L) = 0$.

M2: $\mathrm{Ext}^1_R(L, L^*) \neq 0$. In fact, it's generated by $\binom{n}{2}$ elements.

Theorem: $\mathrm{Hom}_R(M, L^*)$ is a MCM R-module, generated by the *alternating* $n \times n$ matrices (so is $\binom{n}{2}$-generated).

(Recall $M = \mathrm{cok}(\mathrm{adj}(X))$, $L = \mathrm{cok}(X)$.)

Cor: For each alternating $n \times n$ matrix A, there is a unique alternating matrix B so that

$$A\,\mathrm{adj}(X) = X^T B$$

If n is *even*, then A can be chosen invertible, so that

$$\mathrm{adj}(X) = A^{-1} X^T B$$

is a nontrivial factorization of $\mathrm{adj}(X)$.

For $A \neq A'$ alternating, invertible, the corresponding extensions in $\mathrm{Ext}^1_R(L, L^*)$ have nonisomorphic middle terms.

This answers Bergman's question: There are *lots* of factorizations. For A not necessarily invertible, we can say more:

Prop: $\operatorname{Ext}^1_R(L, L^*)$ is perfect of grade 4 over $k[x_{ij}]$, $\binom{n}{2}$-generated.

Cor: There is an unbounded family of rank-two, orientable MCM R-modules

$$Q \cong \operatorname{cok} \begin{bmatrix} X & A \\ 0 & X^T \end{bmatrix},$$

for A an alternating $n \times n$ matrix, with $n \leq \mu(Q) \leq 2n$.