Non-commutative desingularizations of determinantal varieties

joint work with R.-O. Buchweitz and M. Van den Bergh

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0. This file is an experiment in using beamer's "note" technology. I've inserted the notes I made to myself when I gave the lecture; each page of notes follows the slide it's attached to. This effectively doubles the size of the file, but might help any readers know what I was saying as the slides were displayed. I hope it's worthwhile.
1. The linebreak in the title is significant: the first half of the talk is meant to be about non-commutative desingularizations in general (or, rather, one potential definition of what a non-commutative desingularization should be in general), while the second half is about determinantal varieties.

Outline

- Definition
- Motivating Examples
- History, Related Results, and Previous Work
- The Generic Determinantal Hypersurface
- Maximal Minors

Definition (Van den Bergh '04)

Let R be a normal Gorenstein domain. A non-commutative crepant resolution of R (or of $\operatorname{Spec} R$) is an R-algebra $\mathcal{E} = \operatorname{End}_R(M)$ such that

- M is a finitely generated reflexive R-module;
- $\mathcal{E}$ has finite global dimension; and
- $\mathcal{E}$ is a maximal Cohen–Macaulay R-module.

1. I spent a long time on this slide, defining all the terms in the bullet points ("reflexive," "finite global dimension," "maximal Cohen-Macaulay").
2. Insisting that M be reflexive is mostly just to rule out degenerate cases like the residue field. So it's sensible to make M torsion-free, and then $\mathrm{End}(M) = \mathrm{End}(M^{**})$, so reflexivity is free.
3. We'll return to the issue of why this is a reasonable definition (and name) later.

Example (Quotient Singularities)

Let $S = \mathbb{C}[\![x_1, \ldots, x_n]\!]$ and $G \subseteq \mathrm{SL}_n(\mathbb{C})$ a finite group acting on S by linear changes of variables. Assume G contains no pseudo-reflections. Set $R = S^G$, a normal Gorenstein domain.

For a particular example, take $n = 2$, $R = \mathbb{C}[\![x^2, xy, y^2]\!]$. Then as R-modules,

$$S \cong R \oplus (x, y)\,.$$

In general, S is always a maximal Cohen–Macaulay module. (In dimension 2, this is the same as "reflexive.")

1. A pseudo-reflection is an element of finite order that fixes a hyperplane. Watanabe proved that as long as G contains no pseudo-reflections, S^G is Gorenstein iff $G \subseteq \mathrm{SL}$. Also worth mentioning: the Shephard-Todd-Chevalley theorem.
2. Recall that MCM implies reflexive for Gorenstein rings, and the converse is true in dimension 2 (though not generally).

Example (Quotient Singularities, cont.)

$S = \mathbb{C}[\![x_1, \dots, x_n]\!]$; $G \subseteq \mathrm{SL}_n(\mathbb{C})$ finite no pseudo-refs; $R = S^G$

Theorem (Auslander '62)

The endomorphism ring $\mathcal{E} = \mathrm{End}_R(S)$ is isomorphic to the twisted group ring $S\#G$, and thus has global dimension n.

In particular, $\mathcal{E}$ is isomorphic over R to a direct sum of copies of S, so is MCM. Thus $\mathcal{E}$ is a non-commutative crepant resolution of R.

Of course, much more is true (dimension 2, Herzog, McKay Correspondence, ...).

1. The "twisted group ring" $S\#G$ is the free S-module on the elements of G, with multiplication "twisted" by the action: moving g past s applies g to s.
2. Herzog proved in '78 that in the case $n = 2$, the indecomposable finitely generated reflexive R-modules are precisely the indecomposable R-direct summands of S. In particular, there are only finitely many up to isomorphism.
3. The McKay correspondence in case $n = 2$ is a one-one correspondence between representations of G, MCM R-modules, and exceptional curves in the resolution (which also extends to graphs built from these data).

Example (2×2 Determinant)

Let $S = \mathbb{C}[x, y, z, w]$, and put $R = S/(xw - zy)$, another normal Gorenstein domain, of dimension 3.

The hypersurface $\operatorname{Spec} R$ has two resolutions of singularities, obtained by blowing up either of the height-one primes

$$I = (x, y) \qquad \text{or} \qquad J = (x, z)\,.$$

We can also think of $\operatorname{Spec} R$ as a hypersurface in

$$\operatorname{Spec} S \cong \left\{ 2 \times 2 \text{ matrices } A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \right\}\,.$$

1. This example is actually a baby case of one of the Main Theorems coming later.
2. Note that I and J are reflexive, even MCM modules, and that in fact $J = I^{-1}$ (in the class group, say).

Example (2×2 cont.)

The blowups at $I = (x, y)$ and $J = (x, z)$ are special cases of the Springer resolutions:

$$\mathcal{Z} = \left\{ \left(A, \begin{bmatrix} a \\ b \end{bmatrix} \right) \in \operatorname{Spec} S \times \mathbb{P}^1 \; : \; A \begin{bmatrix} a \\ b \end{bmatrix} = 0 \right\}$$

and

$$\mathcal{Z}' = \left\{ \left(A, \begin{bmatrix} a \\ b \end{bmatrix} \right) \in \operatorname{Spec} S \times \mathbb{P}^1 \; : \; \operatorname{image} A \subseteq \operatorname{span} \begin{bmatrix} a \\ b \end{bmatrix} \right\}.$$

We'll see these again later.

1. $\operatorname{Spec} R$ is the locus where the matrix A drops rank. In naive terms, this means either that the kernel gets too big, or the image gets too small (of course they're equivalent). The two desingularizations correspond to these ideas.

2. $\mathscr{Z}$ and $\mathscr{Z}'$ can be defined for any square matrix in exactly analogous ways, and they'll be used later. In fact, they can even be defined for non-square matrices, though then they are distinctly different gadgets.

Example (2×2 cont.)

$S = \mathbb{C}[x, y, z, w]; \quad R = S/(xw - zy); \quad I = (x, y); \quad J = (x, z)$

Set

$$\begin{aligned} \mathcal{E} &= \operatorname{End}_R(R \oplus I) \\ &\cong \begin{pmatrix} R & I \\ J & R \end{pmatrix} \end{aligned}$$

Then

- $\mathcal{E}$ has global dimension 3, and
- $\mathcal{E}$ is maximal Cohen–Macaulay as an R-module,

so $\mathcal{E}$ is a non-commutative crepant resolution of R.

1. The matrix ring should be understood to use the isomorphism $J \cong I^{-1}$.
2. $\mathcal{E}$ is MCM since, as an R-module, it's just $R^2 \oplus I \oplus J$.
3. The global dimension of $\mathcal{E}$ is an easy calculation.

Example (2×2 cont.)

$S = \mathbb{C}[x, y, z, w]; \quad R = S/(xw - zy); \quad I = (x, y); \quad J = (x, z)$

Even more,

- $\mathcal{E} = \operatorname{End}_R(R \oplus I)$ is "more symmetric" than $\mathcal{Z}$ and $\mathcal{Z}'$, as it involves both I and J.
- There are equivalences of derived categories

$$D^b(\operatorname{coh} \mathcal{Z}) \cong D^b(\operatorname{mod} \mathcal{E}) \cong D^b(\operatorname{coh} \mathcal{Z}')\,.$$

This seems to have been known to experts for some time, and also follows from our results below.

1. The first comment is obviously an aesthetic opinion, not mathematics.
2. I'm using $D^b(-)$ for the bounded derived category of either f.g. modules or coherent sheaves, as appropriate.
3. Apparently this example is called the "Atiyah flop," though I just recently learned it's not due to Atiyah, and in fact goes back at least to Krull.

The first explicit appearance of non-commutative resolutions seems to have been in theoretical physics (!).

Resolution of Stringy Singularities by Non-commutative Algebras (Berenstein-Leigh '01)

"Our intention in this paper is to make a general proposal for 'resolving' singularities within non-commutative geometry and to understand the D-branes on these spaces. [...] The algebraic geometry of this commutative algebra will be identified with the target space where closed strings propagate, and the algebraic geometry of the non-commutative algebras will give the resolution of those singularities. In this sense a commutative singularity can be made smooth in a non-commutative sense."

1. I'd love to understand this paper.

Much earlier, however, Auslander defined the related notion of representation dimension:

Definition (Auslander, '71)

Let R be an artin algebra (finite over its center). The *representation dimension* of R is

$$\min_{M} \{\mathrm{gl.\,dim}\,\mathrm{End}_R(M)\}$$

where M runs over all generator-cogenerators..

Thus in particular: a non-commutative crepant resolution of a complete local ring gives finiteness of the representation dimension (suitably redefined for non-artinian rings).

1. For us, generator-cogenerators are modules with both a free summand and an injective-envelope summand.

2. The redefinition for higher Krull dimension: Let T be a complete regular local ring, R a finite R-algebra, and M run over all R-modules which are T-free, have an R-direct summand, and have an ω_R-summand.

A few other facts about representation dimension:

Theorem (Auslander, '71)

An artin algebra of finite representation type has representation dimension equal to 2.

It wasn't until 2003 that O. Iyama proved that the representation dimension of an artin algebra is finite.

This area has been very lively lately. Work of Rouquier, Oppermann, and Bergh has greatly increased the number of examples we understand. Connections have also arisen with cluster algebras (Burban, Keller, Iyama, Reiten, et. al), and so on.

1. There's a lot more to say here, including work by Iyama and me in dimension one, rings of finite CM type by me, Rouquier's examples of rings with representation dimension > 3, and Iyama's talk at this conference (immediately after mine), plus much more.

Why is it a "desingularization"?

$\mathcal{E} = \mathrm{End}_R(M)$	$\mathcal{E}$ is a finitely generated R-module	"proper"
$\mathcal{E} = \mathrm{End}_R(M)$	$\mathcal{E} \otimes_R K \cong M_n(K)$, where K is the quotient field	"birational"
$\mathrm{gl.\,dim}\,\mathcal{E} < \infty$	$\mathcal{E}$ is nonsingular	"smooth"
$\mathcal{E}$ MCM / R	$\mathrm{Ext}_R^{>0}(\mathcal{E}, R) = 0$, so $\mathcal{E}$ is its own dualizing module	"crepant"

1. We're substituting "finitely generated" for "proper". Also, the condition $\mathcal{E} \otimes_R K \cong M_n(K)$ says that $\mathcal{E} \otimes K$ is Morita equivalent to K, which is the point. The last row of the table of course depends heavily on R being Gorenstein.

Theorem (Van den Bergh, '04)

Assume that $\dim R = 3$ *and* $\operatorname{Spec} R$ *has terminal singularities. Then* R *has a non-commutative crepant resolution if and only if it has a commutative crepant resolution.*

A "commutative" crepant resolution is a resolution of singularities $\pi\colon Y \longrightarrow X$ with $\pi^*\omega_X = \omega_Y$. Existence of such a resolution is quite strong: among other things, it forces X to have rational singularities.

Theorem (Stafford-Van den Bergh, '06)

If R *has characteristic zero and possesses a non-commutative crepant resolution of singularities, then it has (at worst) rational singularities.*

1. In some sense, the MCM condition on $\mathcal{E}$ is a way of insisting that $\omega_{\mathcal{E}} = \mathcal{E}$, which is a substitute for $\pi^*\omega_X = \omega_Y$.
2. Van den Bergh had proved the graded case of the Stafford-VdB theorem previously.

Conjecture (Van den Bergh)

All crepant resolutions of $X = \operatorname{Spec} R$ *— commutative as well as non-commutative — have equivalent bounded derived categories.*

In particular, this would imply that $D^b(\operatorname{coh} Y) \cong D^b(\operatorname{coh} Y')$ for all commutative crepant resolutions $Y, Y' \longrightarrow X$, which was conjectured by Bondal and Orlov ('02).

1. In fact, Bondal-Orlov's conjecture was (I believe) Van den Bergh's initial motivation for considering non-commutative crepant resolutions.

Main Theorem 1

Let $X = (x_{ij})$ be the generic square matrix of size $n \geq 2$, $S = k[X]$, and $R = S/\det X$ the generic determinantal hypersurface. Then R has a non-commutative crepant resolution.

1. Of course R is a normal Gorenstein domain. It has dimension $n^2 - 1$. The determinant is a homogeneous polynomial of degree n.
2. The case $n = 2$ was the Example from before; that was an isolated singularity, but for $n \geq 3$, R is very singular – singular in codimension 3. (The reason this is relevant is that examples of non-commutative crepant resolutions of isolated singularities are hard to come by.)

Construction

Consider X as a map between two free S-modules of rank n:

$$0 \longrightarrow \mathcal{G} \xrightarrow{X} \mathcal{F} \longrightarrow M_1 = \operatorname{cok} X \longrightarrow 0\,.$$

Take exterior powers:

$$0 \longrightarrow \bigwedge^k \mathcal{G} \xrightarrow{\bigwedge^k X} \bigwedge^k \mathcal{F} \longrightarrow M_k \longrightarrow 0$$

for $k = 1, \ldots, n$ (in particular $M_n \cong S/\det X = R$).

Fact

Each M_k is a maximal Cohen–Macaulay R-module of rank $\binom{n}{k}$.
This follows from the fact that $\bigwedge^k X$ and its "adjoint" $\bigwedge^{n-k} X^T$ form a matrix factorization of $\det X$.

1. The "matrix factorization" assertion is essentially (a version of) Cramer's Rule, that $\bigwedge^k X \bigwedge^{n-k} X^T = (\det X)\,\mathrm{id}_{\binom{n}{k}}$.
2. The theory of matrix factorizations is of course due originally to Eisenbud.

Main Theorem 1, precise version

Set

$$M = \bigoplus_k M_k = \bigoplus_k \operatorname{cok}\left(\bigwedge^k X\right).$$

and $\mathcal{E} = \operatorname{End}_R(M)$. *Then*

- $\mathcal{E}$ *is MCM over* R*;*
- $\operatorname{Ext}_R^{odd}(M, M) = 0$*;*
- $\mathcal{E}$ *has finite global dimension; and*
- $D^b(\operatorname{mod}\mathcal{E}) \cong D^b(\operatorname{coh}\mathcal{Z})$, *where* $\mathcal{Z}$ *is the Springer desingularization of* $\operatorname{Spec} R$*. (Note that this exists for all* n*, defined analogously to the case* $n = 2$*.)*

1. Note that since R is a hypersurface, Ext is periodic; all the odd ones are isomorphic, as are all the even ones.
2. $\mathrm{Ext}^2(M, M)$ is the stable endomorphism ring $\underline{\mathrm{End}}(M)$ of M (the endomorphism ring modulo those maps factoring through a free module), so is of course not zero.
3. In fact, we have some partial results about the relationship between $\underline{\mathrm{End}}(M)$ and the singular locus of R, but they're not fully baked yet.

How to describe $\mathcal{E} = \operatorname{End}_R(M)$?

- The projectors $e_i \colon M \twoheadrightarrow M_i \hookrightarrow M$ are idempotent endomorphisms.

There are also some "obvious" elements of $\operatorname{Hom}_R(M_i, M_j)$ with $i \neq j$.

Such a homomorphism is a pair (α, β) making

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge^i \mathcal{G} & \xrightarrow{\bigwedge^i X} & \bigwedge^i \mathcal{F} & \longrightarrow & M_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \vdots & & \\ 0 & \longrightarrow & \bigwedge^j \mathcal{G} & \xrightarrow{\bigwedge^j X} & \bigwedge^j \mathcal{F} & \longrightarrow & M_j & \longrightarrow & 0 \end{array}$$

commute.

- Each element $f^* \in \mathcal{F}^*$ defines a contraction

$$\partial_{f^*} \colon \bigwedge^{i} \mathcal{F} \longrightarrow \bigwedge^{i-1} \mathcal{F}$$

which lifts to give a degree -1 endomorphism $M_a \longrightarrow M_{a-1}$.

- Each element $g \in \mathcal{G}$ defines a multiplication

$$\mu_g \colon \bigwedge^{i} \mathcal{G} \longrightarrow \bigwedge^{i+1} \mathcal{G}$$

which gives a degree $+1$ endomorphism $M_a \longrightarrow M_{a+1}$.

Note that there are relations among the ∂_{f^*}, μ_g, and e_i, notably

$$\mu_g \partial_{f^*} + \partial_{f^*} \mu_g = f^*(X(g)) \in S\,.$$

1. ∂ is just the usual map in the Koszul complex on F^*.
2. The so-called "Clifford relation" at the bottom of the page is quadratic in ∂ and μ. We usually think of the exterior algebra as a graded object, so that both sides have degree zero.

Define the “quivered Clifford Algebra”

$$\mathcal{C} := S\langle e_1, \dots, e_n; u_1, \dots, u_n; v_1, \dots, v_n \rangle / \mathcal{J} \,,$$

where $\mathcal{J}$ is the ideal generated by the relations

- $e_a e_b = \delta_{ab} e_a$, $\sum e_a = 1$
- $v_i e_a = e_{a+1} v_i$
- $u_i e_a = e_{a-1} u_i$
- $u_i u_j + u_j u_i = u_i^2 = v_i v_j + v_j v_i = v_i^2 = 0$
- $u_i v_j + v_j u_i = x_{ij}$ (the “Clifford relation”).

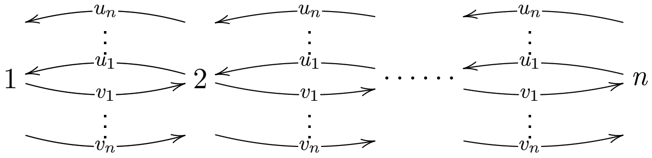
Theorem

$$\mathcal{E} \cong \mathcal{C} \,.$$

1. SLOW! (This slide and the next one!)

Why "quiverized"?

We can realize $\mathcal{C}$ as the path algebra of a quiver with n vertices corresponding to the e_a:



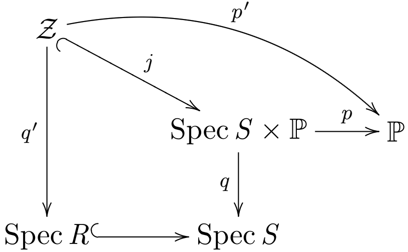
with relations like those defining $\mathcal{C}$.

Proposition

The Springer desingularization $\mathcal{Z}$ is a moduli space of representations of this quiver, subject to a stability condition.

Idea of the Proof

The Springer desingularization comes equipped with projections to $\operatorname{Spec} R$ and $\mathbb{P} = \mathbb{P}^{n-1}$:



1. Time for a confession: all the preceding statements are purely algebraic, but our proofs are all geometric. What we actually do is build a “tilting bundle” on the Springer desingularization $\mathcal{Z}$.
2. The goal is therefore to identify a certain coherent sheaf on $\mathbb{P}$, pull it back to $\mathcal{Z}$, prove it's a tilting sheaf there, then push it down to $\operatorname{Spec} R$ and prove that its direct image is the module M we care about. The vanishing conditions to be a tilting sheaf (which I'm deliberately not saying) follow from some brute-force calculations of cohomology.

Proposition

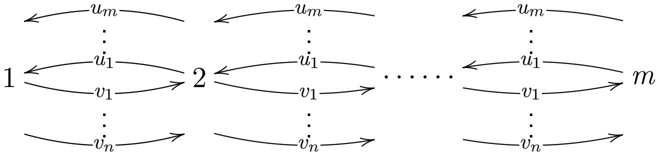
$M_k = \mathcal{R}q_* p^*(\Omega_{\mathbb{P}}^{k-1}(k))$, that is, the direct image of $p^*\Omega^{k-1}(k)$ is M_k and the higher direct images vanish.

This allows us to get an explicit resolution for each $\operatorname{Hom}_R(M_i, M_j)$ and observe that they are MCM. The vanishing further implies the equivalence of derived categories between $\mathcal{E}$ and $\mathcal{Z}$, which forces $\mathcal{E}$ to have finite global dimension.

Main Theorem 2

Let $X = (x_{ij})$ be the generic $m \times n$ matrix with $n \geq m$, $S = k[X]$, and $I_m(X)$ the ideal of maximal minors of X. Then $R = S/I_m(X)$ has a non-commutative desingularization.

In this case, $\mathcal{E}$ is isomorphic to the path algebra of the quiver



with relations

$$u_i u_j + u_j u_i = 0 = u_i^2;$$
$$v_i v_j + v_j v_i = 0 = v_i^2;$$
$$u_k(u_i v_j + v_j u_i) = (u_i v_j + v_j u_i)u_k; \text{ and}$$
$$v_l(u_i v_j + v_j u_i) = (u_i v_j + v_j u_i)v_l.$$

Lots to say here.

1. The ring defined by the maximal minors is not Gorenstein for $n \neq m$, which is why I say "non-commutative desingularization" rather than "non-commutative crepant resolution." We import the definition verbatim (see the first slide, and just delete the word "Gorenstein"). Unfortunately this means we lose many of the nice properties of non-commutative crepant resolutions.
2. The previous statements from the $n = m$ case all still hold. In particular, the module we choose is exactly the same: the direct sum of the cokernels of the exterior powers of the matrix X. These are again known to be MCM (since $n > m$) by Buchsbaum-Rim (or Vetter). Again, the Springer resolution is a moduli space of representations of the quiver. And again the proofs go by way of a tilting bundle on the Springer resolution.