What is a non-commutative desingularization?

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Lincoln — 19 Sept 2008


RESOLUTION
University-wide Award for Outstanding Departmental Teaching
Mathematics and Statistics
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University of Nebraska Board of Regents

RESOLUTION

Whereas, the University-wide Departmental Teaching Award was established in 1992 to recognize an academic department for its outstanding dedication to the education of students; and

Whereas, the criteria for this award include departmental efforts to promote good teaching, to contribute to scholarship on instruction, to provide effective student advising, to provide quality programs for both majors and non-majors, and to engage in productive outreach efforts; and

Whereas, this award carries with it funding in the amount of $25,000 to be used in any manner the department believes appropriate for the further improvement of its instructional programs; and

Whereas, the 1998 winner of the University-wide Departmental Teaching Award is the Department of Mathematics and Statistics, University of Nebraska-Lincoln;

Now, therefore, be it resolved, that the University of Nebraska Board of Regents congratulates and commends the faculty and staff of the University of Nebraska-Lincoln Department of Mathematics and Statistics, and wishes them continued success on offering superlative education and support services to students.

May 16, 1998

Corporation Secretary

University of Nebraska-Lincoln University of Nebraska Medical Center University of Nebraska at Omaha University of Nebraska at Kearney

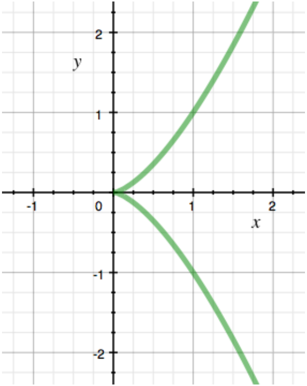
What is a Resolution of Singularities?

Suppose X is an algebraic variety, meaning the solution set of a system of polynomial equations. In general, X might have singularities, or non-smooth points, which prevent it from being a manifold.

Definition (Ver. 1)

A resolution of singularities of X is a parametrization of X by the points of a manifold.

Example: the cusp



Consider the cusp $C\colon \{x^3 = y^2\}$.

We can parametrize C by the points of a line: $t \mapsto (t^2, t^3)$.

Not only is this a parametrization, it's a smooth map (almost) everywhere, in the sense that the derivatives don't simultaneously vanish.

Another way to think of resolving this singularity:

Technically speaking, this is the parametrization $t \mapsto (t^2, t^3, t)$.
(This corresponds to the "blowup algebras" coming later.)

More precisely...

Definition (Ver. 2)

A resolution of singularities is a map of varieties $\pi\colon \widetilde{X} \longrightarrow X$ where

- $\widetilde{X}$ is a manifold;
- π is a surjective map;
- π is differentiable, and almost everywhere a diffeomorphism;
- π is *proper*, in the sense that $\pi^{-1}(\text{compact})$ is compact.

Definition (Ver. 3, Algebra)

Let R be an integral domain with quotient field Q. A resolution of singularities of R is an intermediate ring $R \subseteq A \subseteq Q$ such that A is regular.

The condition that A is regular means precisely that A corresponds to a smooth variety. Generally speaking, we can think of A as a polynomial (or power series) ring over $\mathbb{C}$.

An extension of domains $R \subseteq A$ that share the same quotient field is called *birational*. Geometrically, it means that the corresponding varieties have the same field of rational functions.

The cusp again

The cusp C corresponds to the ring $R = \mathbb{C}[x,y]/(x^3 - y^2)$. Using the parametrization we've already seen, this is isomorphic to $\mathbb{C}[t^2, t^3]$ with quotient field $\mathbb{C}(t)$, the full field of rational functions. We can thus take

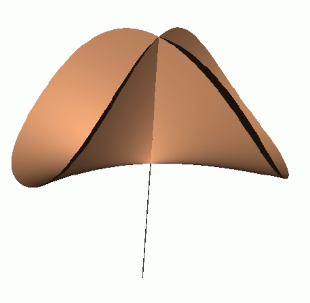
$$A = \mathbb{C}[t]\,.$$

In this case, all we did was "normalize". More on that later.

Whitney Umbrella

Let $W \colon \{x^2 = y^2 z\}$, or equivalently

$$R = \mathbb{C}[x, y, z]/(x^2 - y^2 z)\,.$$



This time it's much less obvious (though easy to check) that we can parametrize W by

$$(s, t) \mapsto (s, t, s^2)$$

or with

$$A = R\left[\frac{x}{y}\right] = \mathbb{C}\left[\frac{x}{y}, y, z\right] / \left(\left(\frac{x}{y}\right)^2 - z\right)\,.$$

An important change to the algebraic definition

In general, the requirement that $A \subseteq Q$ is too strong. We need to allow blowup algebras

$$A = R[It]$$

for ideals I of R. Geometrically, these correspond to "gluing together" birational extensions, though the result is no longer birational. In particular, they are almost never finitely generated as R-modules.

A blowup algebra should be considered smooth if its constituent pieces are — *but $R[It]$ may not be a polynomial ring itself.* Instead, it is smooth "on the punctured spectrum."

Why resolve singularities?

Nonsingular varieties are *much* easier to work with than singular ones, and the definition is built to transmit the information from the resolution to the resolved variety.

Many technical results in algebraic geometry, for example on vanishing of cohomology (Kodaira Vanishing, Kawamata-Viehweg Vanishing, . . .) are proved in exactly this way.

Other examples come from differential equations, dynamical sysytems, etc.

Can it be done? Yes, *usually*

Curves

Normalization (a.k.a. integral closure) resolves singularities of curves. Known to Italian geometers in the 19th century; proven carefully by (at least) Kronecker and Max Noether.

In this case no blowup algebras are needed; one really can find $A \subseteq Q$.

Surfaces defined over $\mathbb{C}$

Walker (1936) and Zariski (1939). Now blowup algebras are really required.

3-folds defined over $\mathbb{C}$

Zariski (1944)

Surfaces or 3-folds in char. p

Abhyankar (1956, 1966)

Can it be done?

Any variety over $\mathbb{C}$

Hironaka (1964)

The proof is 200 pages, and consists (among other things) of 14 nested inductions. It's been called the most complicated mathematical work in history.

The proof has been streamlined somewhat since then, notably by Villamayor, Bierstone–Milman, Encinas–Hauser. It's also been made constructive, which the original proof was not. Still, it remains difficult.

Some petty complaints

Constructing a resolution of singularities of a given variety . . .

- is generally very complicated.
- uses a great deal of geometric machinery.
- introduces blowup algebras and their punctured spectra, which are difficult to analyze.
- can't be done (yet) over fields of prime characteristic or the integers.

Naive hope

Is there a construction that has the same applications, but ...

- avoids the blowup algebras, possibly sticking to module-finite R-algebras?
- works over any ground ring?

No. For example, the "trumpet" $\mathbb{C}[x,y,z]/(x^3+y^2+z^2)$ has no regular finite extensions inside its quotient field.

A potential solution: allow non-commutative A

This also implicates a change in philosophy: we'll focus on *homological aspects of the modules* over A, rather than ring-theoretic properties.

In particular, we'll think of two rings as "the same" if they have equivalent module categories, specifically if they're Morita equivalent:

$$A \simeq_{\text{Morita}} B\,.$$

In particular this holds if

$$B \cong \operatorname{End}_A(A^n)$$

for some free A-module A^n.

The first explicit appearance of non-commutative resolutions seems to have been in theoretical physics (!).

Resolution of Stringy Singularities by Non-commutative Algebras (Berenstein-Leigh '01)

"Our intention in this paper is to make a general proposal for 'resolving' singularities within non-commutative geometry and to understand the D-branes on these spaces. [...] The algebraic geometry of this commutative algebra will be identified with the target space where closed strings propagate, and the algebraic geometry of the non-commutative algebras will give the resolution of those singularities. In this sense a commutative singularity can be made smooth in a non-commutative sense."

Birationality and regularity

Problem

It makes no sense to insist that $A \subseteq Q$ anymore, or even that A is gotten by gluing together such rings, as that would force A to be commutative.

Solution

We force A to sit inside some ring Morita equivalent to Q. Since Q is a field, this means

$$A \subseteq \mathrm{Mat}_n(Q)$$

for some n.

We also now insist A is finitely generated over R.

Birationality and Regularity II

Question

What does "regular" or "smooth" mean for non-commutative rings?

Solution

Translating into homological algebra, this should mean finite global dimension, every module having finite projective dimension.

Warning: This condition is *much weaker* for non-commutative rings than for commutative ones. We'll need to impose extra conditions to have a functional theory.

In particular, we should at least assume A is homologically homogeneous: every simple A-module has the same projective dimension. Even this isn't quite enough.

Crepant resolutions

The most hospitable resolutions of singularities are the crepant ones. A resolution $\pi\colon \widetilde{X} \longrightarrow X$ is crepant if the *canonical bundles* are isomorphic:

$$\pi^*\omega_X = \omega_{\widetilde{X}}\,.$$

This has good technical implications for the transmission of information from $\widetilde{X}$ to X.

Existence of a crepant resolution is quite strong: among other things, it forces X to have rational singularities, and the corresponding ring R to be a Gorenstein normal domain.

Finally, the definition

Definition (Van den Bergh)

Let R be a Gorenstein normal domain. A non-commutative crepant resolution of singularities of R is a homologically homogeneous ring Λ of the form

$$\Lambda = \mathrm{End}_R(M)$$

for M a finitely generated reflexive R-module.

Equivalently, $\Lambda = \mathrm{End}_R(M)$ is a non-commutative crepant resolution if

- Λ has finite global dimension;
- Λ is maximal Cohen–Macaulay as an R-module.

Three justifying theorems

Theorem (Stafford-Van den Bergh, '06)

If R has characteristic zero and possesses a non-commutative crepant resolution of singularities, then it has (at worst) rational singularities.

Theorem (Van den Bergh, '04)

Assume that $\dim R = 3$ and $\operatorname{Spec} R$ has terminal singularities. Then R has a non-commutative crepant resolution if and only if it has a commutative crepant resolution.

Corollary (Conjecture of Bondal and Orlov)

In the above situation, any two commutative crepant resolutions have equivalent derived categories.

The generic determinant

Theorem (Buchweitz-Leuschke-Van den Bergh)

Let $X = (x_{ij})$ be the generic square matrix of size $n \geq 2$, $S = k[X]$, and $R = S/\det X$ the generic determinantal hypersurface. Then R has a non-commutative crepant resolution.

Construction

Consider X as a map between two free S-modules of rank n:

$$0 \longrightarrow \mathcal{G} \xrightarrow{X} \mathcal{F} \longrightarrow M_1 = \operatorname{cok} X \longrightarrow 0\,.$$

Take exterior powers:

$$0 \longrightarrow \bigwedge^k \mathcal{G} \xrightarrow{\bigwedge^k X} \bigwedge^k \mathcal{F} \longrightarrow M_k \longrightarrow 0$$

for $k = 1, \ldots, n$ (in particular $M_n \cong S/\det X = R$).

Fact

Each M_k is a maximal Cohen–Macaulay R-module of rank $\binom{n}{k}$.
This follows from the fact that $\bigwedge^k X$ and its "adjoint" $\bigwedge^{n-k} X^T$ form a matrix factorization of $\det X$.

Generic determinant, precise version

Theorem

Set

$$M = \bigoplus_k M_k = \bigoplus_k \operatorname{cok}\left(\bigwedge^k X\right).$$

and $\mathcal{E} = \operatorname{End}_R(M)$. *Then*

- $\mathcal{E}$ *is MCM over* R*;*
- $\operatorname{Ext}_R^{odd}(M, M) = 0$*;*
- $\mathcal{E}$ *has finite global dimension; and*
- $D^b(\operatorname{mod}\mathcal{E}) \cong D^b(\operatorname{coh}\mathcal{Z})$, *where* $\mathcal{Z}$ *is the Springer desingularization of* $\operatorname{Spec} R$.

How to describe $\mathcal{E} = \mathrm{End}_R(M)$?

- The projectors $e_i \colon M \twoheadrightarrow M_i \hookrightarrow M$ are idempotent endomorphisms.

There are also some "obvious" elements of $\mathrm{Hom}_R(M_i, M_j)$ with $i \neq j$.

Such a homomorphism is a pair (α, β) making

$$\begin{tikzcd}
0 \arrow[r] & \bigwedge^i \mathcal{G} \arrow[r, "\bigwedge^i X"] \arrow[d, "\alpha"'] & \bigwedge^i \mathcal{F} \arrow[r] \arrow[d, "\beta"] & M_i \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \bigwedge^j \mathcal{G} \arrow[r, "\bigwedge^j X"] & \bigwedge^j \mathcal{F} \arrow[r] & M_j \arrow[r] & 0
\end{tikzcd}$$

commute.

- Each element $f^* \in \mathcal{F}^*$ defines a contraction

$$\partial_{f^*} \colon \bigwedge^{i} \mathcal{F} \longrightarrow \bigwedge^{i-1} \mathcal{F}$$

  which lifts to give a degree -1 endomorphism $M_a \longrightarrow M_{a-1}$.
- Each element $g \in \mathcal{G}$ defines a multiplication

$$\mu_g \colon \bigwedge^{i} \mathcal{G} \longrightarrow \bigwedge^{i+1} \mathcal{G}$$

  which gives a degree $+1$ endomorphism $M_a \longrightarrow M_{a+1}$.

Note that there are relations among the ∂_{f^*}, μ_g, and e_i, notably

$$\mu_g \partial_{f^*} + \partial_{f^*} \mu_g = f^*(X(g)) \in S\,.$$

The "quiverized Clifford Algebra"

Theorem

$$\mathcal{E} \cong S\langle e_1, \ldots, e_n; u_1, \ldots, u_n; v_1, \ldots, v_n\rangle / \mathcal{J} \,,$$

where $\mathcal{J}$ is the ideal generated by the relations

- $e_a e_b = \delta_{ab} e_a$, $\sum e_a = 1$
- $v_i e_a = e_{a+1} v_i$
- $u_i e_a = e_{a-1} u_i$
- $u_i u_j + u_j u_i = u_i^2 = v_i v_j + v_j v_i = v_i^2 = 0$
- $u_i v_j + v_j u_i = x_{ij}$ *(the "Clifford relation").*

Why "quiverized"?

We can realize $\mathcal{E}$ as the path algebra of a quiver with n vertices corresponding to the e_a:

$$\begin{tikzcd}[column sep=huge]
1 \arrow[r, bend right=20, "v_1"'] \arrow[r, bend right=50, "v_n"'] & 2 \arrow[l, bend right=20, "u_1"'] \arrow[l, bend right=50, "u_n"'] \arrow[r, bend right=20, "v_1"'] \arrow[r, bend right=50, "v_n"'] & \cdots\cdots \arrow[l, bend right=20, "u_1"'] \arrow[l, bend right=50, "u_n"'] \arrow[r, bend right=20, "v_1"'] \arrow[r, bend right=50, "v_n"'] & n \arrow[l, bend right=20, "u_1"'] \arrow[l, bend right=50, "u_n"']
\end{tikzcd}$$

with relations like those defining $\mathcal{C}$.

Idea of the Proof

The Springer desingularization comes equipped with projections to $\operatorname{Spec} R$ and $\mathbb{P} = \mathbb{P}^{n-1}$:

$$\begin{tikzcd}
\mathcal{Z} \arrow[dr, hook, "j"] \arrow[drr, bend left, "p'"] \arrow[dd, "q'"'] & & \\
& \operatorname{Spec} S \times \mathbb{P} \arrow[r, "p"] \arrow[d, "q"'] & \mathbb{P} \\
\operatorname{Spec} R \arrow[r, hook] & \operatorname{Spec} S &
\end{tikzcd}$$

Proposition

$M_k = \mathcal{R}q_*p^*(\Omega_{\mathbb{P}}^{k-1}(k))$, *that is, the direct image of* $p^*\Omega^{k-1}(k)$ *is* M_k *and the higher direct images vanish.*

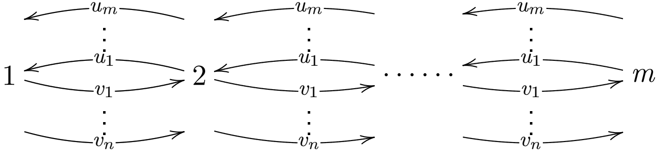
This allows us to get an explicit resolution for each $\mathrm{Hom}_R(M_i, M_j)$ and observe that they are MCM. The vanishing further implies the equivalence of derived categories between $\mathcal{E}$ and $\mathcal{Z}$, which forces $\mathcal{E}$ to have finite global dimension.

Maximal Minors

Theorem

Let $X = (x_{ij})$ be the generic $m \times n$ matrix with $n \geq m$, $S = k[X]$, and $I_m(X)$ the ideal of maximal minors of X. Then $R = S/I_m(X)$ has a non-commutative desingularization.

In this case, $\mathcal{E}$ is isomorphic to the path algebra of the quiver



with relations

$$u_i u_j + u_j u_i = 0 = u_i^2;$$
$$v_i v_j + v_j v_i = 0 = v_i^2;$$
$$u_k(u_i v_j + v_j u_i) = (u_i v_j + v_j u_i)u_k; \text{ and}$$
$$v_l(u_i v_j + v_j u_i) = (u_i v_j + v_j u_i)v_l.$$

Thank you!