Non-commutative desingularization of the generic determinant

joint work with R.-O. Buchweitz and M. Van den Bergh

Graham J. Leuschke

Syracuse University

30 April 2006

Recall (part of) the McKay Correspondence: Let $G \subset \mathrm{GL}_2(\mathbb{C})$ be a finite group, acting on $S = \mathbb{C}[\![x, y]\!]$ by linear changes of variables. Set $R = S^G$, the ring of invariants.

Assume that G contains no pseudo-reflections.

Theorem (Herzog '78, Auslander '86)

There are one-one correspondences between

- *the irreducible representations of G*
- *the indecomposable projective $\mathrm{End}_R(S)$-modules*
- *the indecomposable reflexive R-modules.*

Furthermore, the indecomposable reflexive R-modules are precisely the R-direct summands of S. There are in particular only finitely many indecomposable MCM R-modules.

$$R = S^G \subset S = \mathbb{C}[\![x, y]\!]$$

Proposition (Auslander '62)

The endomorphism ring $\mathrm{End}_R(S)$ *has global dimension equal to* 2*.*
(In fact, it is isomorphic to the twisted group algebra $S\#G$*.)*

Since

$$S \cong \bigoplus_{\substack{_R M \text{ is MCM} \\ M \text{ indec.}}} M^a\,,$$

we have:

> *The endomorphism ring of the direct sum of all the indecomposable MCM R-modules has finite global dimension.*

Definition

Let R be a Gorenstein local domain. A non-commutative crepant resolution of R is an R-algebra $\Gamma = \mathrm{End}_R(M)$ such that

- M is a reflexive R-module
- Γ is a MCM R-module
- $\mathrm{gl.\,dim.}\,\Gamma < \infty$.

Corollary

A Gorenstein two-dimensional complete local $\mathbb{C}$-algebra of finite CM type has a non-commutative crepant resolution.

Theorem (GL)

Let R be a CM local ring of finite CM type, and let M be the direct sum of all indecomposable MCM modules. Then $\mathrm{End}_R(M)$ has global dimension equal to $\dim R$ (but need not be MCM!).

Main Theorem

The generic determinantal hypersurface has a non-commutative crepant resolution, which can be described as a quiverized Clifford algebra. The standard Springer desingularization is obtained as a moduli space of representations of of this algebra.

Notation

- K = field
- $X = (x_{ij})$, a $(n \times n)$ matrix of indeterminates
- $S = K[x_{ij}]$
- $R = S/(\det X)$
- $\mathcal{G}, \mathcal{F}$ free S-modules of rank n
- $\varphi = (x_{ij}) \colon \mathcal{G} \longrightarrow \mathcal{F}$ the generic linear map

It's known that R does not have finite CM type if $n > 2$. In fact R has a wild family of rank-two orientable MCM modules (Buchweitz-GL '05).

It does, however, have a nice family of canonical MCM modules.

Define M_1 by

$$0 \longrightarrow \mathcal{G} \xrightarrow{\varphi} \mathcal{F} \longrightarrow M_1 \longrightarrow 0$$

and

$$0 \longrightarrow \bigwedge^k \mathcal{G} \xrightarrow{\bigwedge^k \varphi} \bigwedge^k \mathcal{F} \longrightarrow M_k \longrightarrow 0$$

for every $k = 0, \dots, n$.

Since $(\bigwedge^k \varphi)(\bigwedge^{n-k} \varphi^*) = (\det \varphi) \cdot I_{\binom{n}{k}}$, these are all MCM.

Main Theorem, precise version

Set

$$M = \bigoplus_k M_k = \bigoplus_k \operatorname{cok}\left(\bigwedge^k \varphi\right) .$$

Then

- $\operatorname{End}_R(M)$ *is MCM over* R*;*
- $\operatorname{Ext}_R^{odd}(M, M) = 0$*;*
- $\operatorname{End}_R(M)$ *has finite global dimension; and*
- $D^b(\operatorname{mod}\operatorname{End}_R(M)) \cong D^b(\operatorname{coh}\mathcal{Z})$, *where* $\mathcal{Z}$ *is some desingularization of* $\operatorname{Spec} R$*.*

Beĭlinson's "tilting description" of the derived category of projective space

Let $\mathbb{P} = \mathbb{P}^{n-1} = \mathbb{P}(V)$. Consider two families of n locally free sheaves on $\mathbb{P}$:

- $\{\mathcal{O}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(-1), \ldots, \mathcal{O}_{\mathbb{P}}(-n+1)\}$
- $\{\mathcal{O}_{\mathbb{P}}(1), \Omega^1(2), \ldots, \Omega^{n-1}(n)\}$ where Ω^1 is the sheaf of Kähler differentials.

Theorem (Beĭlinson '78)

Let

$$\mathcal{E}_1 = \mathrm{End}_{\mathbb{P}}\Big(\bigoplus_{i=0}^{n-1} \mathcal{O}_{\mathbb{P}}(-i)\Big) \qquad \textit{and} \qquad \mathcal{E}_2 = \mathrm{End}_{\mathbb{P}}\Big(\bigoplus_{i=0}^{n-1} \Omega^i(i+1)\Big).$$

There are exact equivalences of triangulated categories:

$$D^b(\mathrm{mod}\, \mathcal{E}_1) \leftarrow D^b(\mathrm{coh}\, \mathbb{P}) \longrightarrow D^b(\mathrm{mod}\, \mathcal{E}_2)$$

Key Lemma

We have $\mathrm{Ext}^{\ell}_{\mathbb{P}}(\Omega^i(i+1), \Omega^j(j+1)) = 0$ *for all* $\ell > 0$. *Furthermore,*

$$\mathrm{Hom}_{\mathbb{P}}(\Omega^i(i+1), \Omega^j(j+1)) \cong \begin{pmatrix} K & V^* & \dots & \dots & \Lambda^{n-1}(V^*) \\ 0 & K & V^* & \dots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & K & V^* \\ 0 & 0 & \dots & \dots & K \end{pmatrix}_{ij} .$$

and a similar version for $\mathrm{Hom}_{\mathbb{P}}(\mathcal{O}(-i), \mathcal{O}(-j))$ *involving* $\mathrm{Sym}_{n-1}(V)$.

The Key Lemma can be described in terms of a quiver with relations. The ring

$$\begin{pmatrix} K & V^* & \dots & \dots & \Lambda^{n-1}(V^*) \\ 0 & K & V^* & \dots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & K & V^* \\ 0 & 0 & \dots & \dots & K \end{pmatrix}$$

has n idempotents; take these for vertices. Each basis element of V^* gives an arrow $a \longrightarrow a-1$

$$1 \;\underset{u_1}{\overset{u_n}{\underset{\vdots}{\longleftarrow}}}\; 2 \;\underset{u_1}{\overset{u_n}{\underset{\vdots}{\longleftarrow}}}\; \cdots\cdots \;\underset{u_1}{\overset{u_n}{\underset{\vdots}{\longleftarrow}}}\; n$$

subject to relations $u_i u_j + u_j u_i = 0$.

This is a "quiverized exterior algebra."

Back to $\operatorname{End}_R(M)$, $R = K[X]/(\det X)$, $M = \bigoplus_k M_k$.

There are some canonical elements of $\operatorname{Hom}_R(M_i, M_j)$.

We seek pairs (α, β) making

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigwedge^i \mathcal{G} & \xrightarrow{\bigwedge^i \varphi} & \bigwedge^i \mathcal{F} & \longrightarrow & M_i & \longrightarrow & 0 \\
 & & \Big\downarrow \alpha & & \Big\downarrow \beta & & \vdots & & \\
0 & \longrightarrow & \bigwedge^j \mathcal{G} & \xrightarrow{\bigwedge^j \varphi} & \bigwedge^j \mathcal{F} & \longrightarrow & M_j & \longrightarrow & 0
\end{array}$$

commute.

- Each element $f^* \in \mathcal{F}^*$ defines a contraction

$$\partial_{f^*} \colon \bigwedge^{i} \mathcal{F} \longrightarrow \bigwedge^{i-1} \mathcal{F}$$

which lifts to give a degree -1 endomorphism $M \longrightarrow M$.
- Each element $g \in \mathcal{G}$ defines a multiplication

$$\mu_g \colon \bigwedge^{i} \mathcal{G} \longrightarrow \bigwedge^{i+1} \mathcal{G}$$

which gives a degree $+1$ endomorphism $M \longrightarrow M$.
- The projectors $e_i \colon M \twoheadrightarrow M_i \hookrightarrow M$ are idempotent endomorphisms of degree 0.

Note that there are relations among the ∂_{f^*}, μ_g, and e_i, notably

$$\mu_g \partial_{f^*} + \partial_{f^*} \mu_g = f^*(\varphi(g)) \in S\,.$$

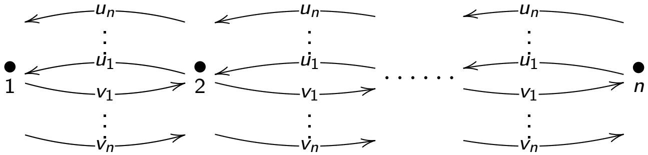
Define the "quiverized Clifford Algebra"

$$\mathcal{C} := S\langle e_1, \dots, e_n; u_1, \dots, u_n; v_1, \dots, v_n\rangle / \mathcal{J} \ ,$$

where $\mathcal{J}$ is the ideal generated by the relations

- $e_a e_b = \delta_{ab} e_a$, $\sum e_a = 1$
- $v_i e_a = e_{a+1} v_i$
- $u_i e_a = e_{a-1} u_i$
- $u_i u_j + u_j u_i = u_i^2 = v_i v_j + v_j v_i = v_i^2 = 0$
- $u_i v_j + v_j u_i = x_{ij}$ (the "Clifford relation").

We realize $\mathcal{C}$ as the path algebra of a quiver with vertices corresponding to the e_a:



with relations given as above.

Theorem
$\mathcal{C}$ is a MCM R-module.

Theorem
$\mathrm{End}_R(M) \cong \mathcal{C}$.

Theorem
$\underline{\mathrm{End}}_R(M) \cong \mathcal{C}/\mathcal{C}e_n\mathcal{C}$.

Idea of the Proof.

Let $\mathcal{Z}$ be the classical desingularization of $X = \operatorname{Spec} R$:

$$\begin{aligned} \mathcal{Z} &= \{([\lambda], \alpha) \mid \lambda\alpha = 0\} \\ &\subseteq \mathbb{P}(F^*) \times \operatorname{Hom}_K(G, F) \end{aligned}$$

with projections p to $\mathbb{P}(F^*)$ and q to $\operatorname{Spec} R$.

Proposition

$M_k = \mathcal{R}q_* p^*(\Omega_{\mathbb{P}}^{k-1}(k))$, that is, the direct image of $p^*\Omega^{k-1}(k)$ is M_k and the higher direct images vanish.

Get an explicit resolution for each $\operatorname{Hom}_R(M_i, M_j)$ and observe that they are MCM.

□