Non-commutative desingularization of the generic determinant

joint work with R.-O. Buchweitz and M. Van den Bergh

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7 October 2007

An Extended Example

Fix a field k, and let $S = k[x, y, u, v]$. Consider

$$R = S/(xy - uv) \qquad \text{and} \qquad X = \operatorname{Spec} R\,,$$

a hypersurface in $\operatorname{Spec} S \cong M_2(k)$.

X has "canonical" resolution(s) of singularities

$$\mathcal{Z} = \left\{ \left(\Phi, \begin{bmatrix} a \\ b \end{bmatrix} \right) \in M_2(k) \times \mathbb{P}^1 \ : \ \Phi \begin{bmatrix} a \\ b \end{bmatrix} = 0 \right\}$$

and

$$\mathcal{Z}' = \left\{ \left(\Phi, \begin{bmatrix} a \\ b \end{bmatrix} \right) \in M_2(k) \times \mathbb{P}^1 \ : \ \operatorname{image} \Phi \subseteq \operatorname{span} \begin{bmatrix} a \\ b \end{bmatrix} \right\}\,,$$

sometimes called the Springer resolutions.

Some complaints about $\mathcal{Z}$ and $\mathcal{Z}'$

- They're canonical, but canonically asymmetric.
- We leave our original category, not once but twice.

A proposed replacement for $\mathcal{Z}$ and $\mathcal{Z}'$

Let $I = (x, u)$, a height-one prime of R, and set

$$\begin{aligned} \mathcal{E} &= \mathrm{End}_R(R \oplus I) \\ &\cong \begin{pmatrix} R & I \\ I^{-1} & R \end{pmatrix} \end{aligned}$$

where $I^{-1} = \mathrm{Hom}_R(I, R) \cong (x, v)$.

Note that $\mathcal{Z}$ is the blowup of X at I, while $\mathcal{Z}'$ is the blowup at I^{-1}.

Some observations about $\mathcal{E} = \mathrm{End}_R(R \oplus I)$

- $\mathcal{E}$ is non-commutative!
- $\mathcal{E}$ is manifestly "symmetric", since I is invertible.
- $\mathcal{E}$ is a finitely generated R-module (*"proper"*)
- $\mathrm{gl.\,dim}\,\mathcal{E} = 3$ (*"smooth"*)
- $\mathcal{E} \otimes_R K \cong M_2(K)$ is Morita equivalent to the quotient field K of R (*"birational"*)
- $\mathcal{E}$ is maximal Cohen–Macaulay as an R-module. More to the point, $\mathrm{Ext}^i_R(\mathcal{E}, R) = 0$ for $i > 0$. It follows (since R is Gorenstein) that $\mathcal{E}$ is its own dualizing module, so in particular $\mathcal{E} \otimes_R \omega_R = \omega_{\mathcal{E}}$. (*"crepant"*)

Definition (Van den Bergh)

Let R be a Gorenstein local normal domain. A non-commutative crepant resolution of R is an R-algebra $\mathcal{E} = \mathrm{End}_R(M)$ such that

- M is a reflexive R-module
- $\mathrm{gl.dim}\,\mathcal{E} < \infty$.
- $\mathcal{E}$ is a MCM R-module

Remark

Van den Bergh and Stafford have recently showed that existence of such an $\mathcal{E}$ implies that R has at worst rational singularities (in characteristic zero).

Another source of examples: the McKay correspondence. If $G \subseteq \mathrm{SL}_n(k)$ is a finite group acting on $S = k[x_1, \ldots, x_n]$ with invariant ring $R = S^G$, then $\mathrm{End}_R(S) \cong S\#G$ is a non-commutative crepant resolution of R.

Main Theorem

Let $X = (x_{ij})$ *be the generic square matrix of size* $n \geq 2$*,* $S = k[X]$*, and* $R = S/\det X$*. Then* R *has a non-commutative crepant resolution.*

Construction

Consider X as a map between two free S-modules of rank n:

$$0 \longrightarrow \mathcal{G} \xrightarrow{X} \mathcal{F} \longrightarrow M_1 = \operatorname{cok} X \longrightarrow 0\,.$$

Take exterior powers:

$$0 \longrightarrow \bigwedge^k \mathcal{G} \xrightarrow{\bigwedge^k X} \bigwedge^k \mathcal{F} \longrightarrow M_k \longrightarrow 0$$

for $k = 1, \ldots, n$ (in particular $M_n = R$).

Fact

Each M_k is a maximal Cohen–Macaulay R-module of rank $\binom{n}{k}$.
This follows from the fact that $\bigwedge^k X$ and its “adjoint” $\bigwedge^{n-k} X^T$ form a matrix factorization of $\det X$.

Main Theorem, precise version

Set

$$M = \bigoplus_k M_k = \bigoplus_k \operatorname{cok}\left(\bigwedge^k X\right).$$

Then

- $\operatorname{End}_R(M)$ *is MCM over R;*
- $\operatorname{Ext}_R^{odd}(M, M) = 0$*;*
- $\operatorname{End}_R(M)$ *has finite global dimension; and*
- $D^b(\operatorname{mod}\operatorname{End}_R(M)) \cong D^b(\operatorname{coh}\mathcal{Z})$*, where* $\mathcal{Z}$ *is the Springer desingularization of* $\operatorname{Spec} R$*. (Note that this exists for all* n*, defined analagously to the* $n = 2$ *case.)*

How to describe $\mathrm{End}_R(M)$?

- The projectors $e_i \colon M \twoheadrightarrow M_i \hookrightarrow M$ are idempotent endomorphisms.

There are also some "obvious" elements of $\mathrm{Hom}_R(M_i, M_j)$ with $i \neq j$.

Such a homomorphism is a pair (α, β) making

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge^i \mathcal{G} & \xrightarrow{\bigwedge^i X} & \bigwedge^i \mathcal{F} & \longrightarrow & M_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \vdots & & \\ 0 & \longrightarrow & \bigwedge^j \mathcal{G} & \xrightarrow{\bigwedge^j X} & \bigwedge^j \mathcal{F} & \longrightarrow & M_j & \longrightarrow & 0 \end{array}$$

commute.

- Each element $f^* \in \mathcal{F}^*$ defines a contraction

$$\partial_{f^*} \colon \bigwedge^{i} \mathcal{F} \longrightarrow \bigwedge^{i-1} \mathcal{F}$$

which lifts to give a degree -1 endomorphism $M \longrightarrow M$.

- Each element $g \in \mathcal{G}$ defines a multiplication

$$\mu_g \colon \bigwedge^{i} \mathcal{G} \longrightarrow \bigwedge^{i+1} \mathcal{G}$$

which gives a degree $+1$ endomorphism $M \longrightarrow M$.

Note that there are relations among the ∂_{f^*}, μ_g, and e_i, notably

$$\mu_g \partial_{f^*} + \partial_{f^*} \mu_g = f^*(X(g)) \in S\,.$$

Define the "quiverized Clifford Algebra"

$$\mathcal{C} := S\langle e_1, \ldots, e_n; u_1, \ldots, u_n; v_1, \ldots, v_n\rangle / \mathcal{J} \ ,$$

where $\mathcal{J}$ is the ideal generated by the relations

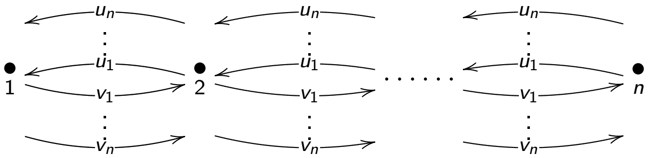
- $e_a e_b = \delta_{ab} e_a$, $\sum e_a = 1$
- $v_i e_a = e_{a+1} v_i$
- $u_i e_a = e_{a-1} u_i$
- $u_i u_j + u_j u_i = u_i^2 = v_i v_j + v_j v_i = v_i^2 = 0$
- $u_i v_j + v_j u_i = x_{ij}$ (the "Clifford relation").

Theorem

$$\mathcal{E} \cong \mathcal{C} \ .$$

Why "quiverized"?

We can realize $\mathcal{C}$ as the path algebra of a quiver with vertices corresponding to the e_a:



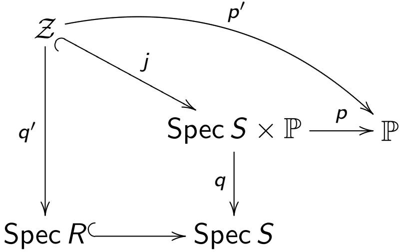
with relations like those defining $\mathcal{C}$.

Proposition

The Springer desingularization $\mathcal{Z}$ is a moduli space of representations of this quiver, subject to a stability condition.

Idea of the Proof.

The Springer desingularization comes equipped with projections to Spec R and $\mathbb{P} = \mathbb{P}^{n-1}$:



Proposition

$M_k = \mathcal{R}q_* p^*(\Omega_{\mathbb{P}}^{k-1}(k))$, that is, the direct image of $p^*\Omega^{k-1}(k)$ is M_k and the higher direct images vanish.

This allows us to get an explicit resolution for each $\mathrm{Hom}_R(M_i, M_j)$ and observe that they are MCM. The vanishing further implies the equivalence of derived categories between $\mathcal{E}$ and $\mathcal{Z}$, which forces $\mathcal{E}$ to have finite global dimension. □