Finite Type:
Recurring Examples

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Contexts:

1. Quivers & Blowups

2. Simplicity

3. Deformations

4. Module Type

The Examples: The curves in $\mathbb{C}^2$ defined by

(A$_n$): $y^2 + x^{n+1} = 0$

(D$_n$): $xy^2 + y^{n-1} = 0$

(E$_6$): $y^3 + x^4 = 0$

(E$_7$): $y^3 + x^3 y = 0$

(E$_8$): $y^3 + x^5 = 0$

Quivers

Def: A *quiver* is a finite directed graph. It can have loops or multiple arrows.

Eg's:

Def: A *representation* of a quiver Q (over $\mathbb{C}$) is an assignment to each vertex x a finite-dimensional vector space $\mathbb{C}^{n_x}$ and to each arrow $x \to y$ a linear map $\mathbb{C}^{n_x} \to \mathbb{C}^{n_y}$.

(We don't assume anything commutes, or agrees, or anything.)

Eg's:

For a fixed quiver Q we can define the direct sum of (finitely many) representations by just taking the direct sum of $\mathbb{C}^{n_x}$ and $\mathbb{C}^{n'_x}$.

There's an obvious notion of *isomorphism* of representations and of *indecomposable* ones (ones that aren't nontrivial direct sums of other ones).

Fact: Every representation is a direct sum of indecomposable ones.

Eg's:

Def: Say a quiver Q has *finite representation type* if there are, up to isomorphism, only finitely many indecomposable representations of Q.

Which quivers have finite representation type?

There are five types (at least if you ignore which way the arrows go):

So we've got 5 (classes of) equations, and 5 (classes of) quivers. How to get from one to the other? Blow'em up.

Def: Let $f(x, y)$ be a polynomial with a singularity at the origin. Define a *blowup* of f at the origin to be a factorization of f using either y/x or x/y.

Eg: Let $f(x, y) = y^2 + x^3$. Then a blowup of f is

$$\begin{aligned} y^2 + x^3 &= x^2 \left((y/x)^2 + x\right) \\ &= x^2(u^2 + x) \end{aligned}$$

The point of this is that, when we think of y/x as a new variable, the blowup now has a factor of the form $x+$something. When these are graphed, they're smooth (no singularities). A factor of the form $y+$something would have been OK too.

The idea of blowing up is that the curve gets smoother.

Sometimes you have to blow up several times to get to a completely smooth curve. The procedure is this: blow up once, factor. Blow up each factor. Repeat. Eventually every factor will have a smooth factor.

Keep track of the factors with a *desingularization diagram*:

Which polynomials have the desingularization diagrams corresponding to the quivers that we saw had finite representation type?

The polynomials $f(x, y)$ whose desingularization quiver has finite representation type are exactly the (A_n), (D_n), (E_6), (E_7), (E_8) polynomials.

Too complicated? Try simplicity

Def: A polynomial $F(x, y)$ is *simple* if there are only finitely many ways to write

$$F = G_1 G_2 + G_3 G_4 + \cdots + G_{2n-1} G_{2n},$$

where "different" means that the g_i generate different *ideals*.

The *ideal* generated by a bunch of polynomials $(G_1, \ldots, G_k)$ is the set of all polynomials

$$G_1 H_1 + \cdots + G_k H_k,$$

where the H_i are polynomials.

Which non-linear polynomials $F(x, y)$ are simple?

(A$_n$): $y^2 + x^{n+1}$

(D$_n$): $xy^2 + y^{n-1}$

(E$_6$): $y^3 + x^4$

(E$_7$): $y^3 + x^3 y$

(E$_8$): $y^3 + x^5$

Maybe that's not surprising enough. Factoring polynomials (blowing up) looks a lot like writing them as $G_1 G_2 + \cdots + G_{2n-1} G_{2n}$.

On the other hand, I don't know of any direct proof that the two concepts are the same.

Something completely different

Def: A *deformation* of a polynomial $f(x, y)$ is a polynomial of the form $f(x, y) + g(x, y)$, where g is not in the ideal of $\mathbb{C}[x, y]$ generated by $(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$.

Notice: That's the Jacobian ideal. Things not in it are "small". Geometrically, this corresponds to twiddling the curve just a little bit.

Def: Say a polynomial $f(x, y)$ has *finite deformation type* if there are only finitely many isomorphism classes of rings $\mathbb{C}[x, y]/(f+g)$, where $f + g$ is a deformation of f.

Which non-linear polynomials have finite deformation type?

You guessed it.

Finite Module Type

Def: Let R be a ring. A *module* M over R is an abelian group, such that elements of R can be multiplied by elements of M. (And satisfy associativity, distributivity, etc.)

We assume all modules are finitely generated.

Eg: Let $R = \mathbb{Z}$. Then R-modules are just abelian groups. The finitely generated ones are direct sums of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$.

Any interesting ring has infinitely many modules. We should restrict the class a little to get anything interesting.

Def: A module M is *torsionfree* if whenever $rm = 0$, where $r \in R$ and $m \in M$, either $m = 0$ or r divides zero in R.

Def: A ring has *finite module type* if it has only finitely many torsionfree modules.

Which rings of the form $\mathbb{C}[x, y]/(f)$ have finite module type?

Take a wild guess.

Bizarre final chapter

You can ask all of these same finiteness questions for polynomials in more than two variables.

The answers are just $f(x, y) + z_1^2 + \cdots + z_n^2$, where f is one of ours and the z_i are new variables.

Crazy.