THE COXETER-DYNKIN DIAGRAMS: A, D, AND E

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ABSTRACT. Certain finite graphs, called the "Coxeter-Dynkin diagrams", seem to come up in every area of pure mathematics. Every time they appear, they reveal deep connections between things as apparently unrelated as Platonic solids, quadratic forms, and representation theory. I'll describe a couple of ways that Coxeter-Dynkin diagrams arise naturally, and hint at some others.

The goal of this talk is to discuss two quite different contexts where the same short list of graphs comes up naturally. In fact there are many many more contexts where the same list appears, usually as the answer to some sort of "positivity" or "finiteness" question. I'll mention a couple more at the end (and Allen Pelley's talk will discuss another).

Outline:

I. Quadratic Forms
II. Quiver Representations

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1. QUADRATIC FORMS

Definition. For today, a *graph* is just a finite collection of *vertices* connected by *edges*. We allow multiple edges and loops, and usually number the vertices.

Examples. (with audience help, some random examples)

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There are many different invariants one might attach to a graph. For example, one might define the *degree* of a vertex to be the number of edges involving that vertex, and associate to the graph its *degree vector.*

Other examples: chromatic number, girth (length of the shortest circuit), etc.

Today we're more interested in an algebraic invariant, the *Tits form* of the graph.

Definition. The *quadratic form* (or *Tits form*, after Jacques Tits) associated to a graph G with n vertices is the polynomial q_G in $x_1, \dots, x_n$ defined by

$$q_G(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^2 - \sum_{i - - j} x_i x_j \, .$$

Note that if i and j have multiple edges between them, we take one monomial for each edge.

Examples.

(1) $1 \text{ --- } 2$ $\qquad q(x_1, x_2) = x_1^2 + x_2^2 - x_1 x_2$

(2) $1 = 2$ $\qquad q(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 x_2$

(3) $1 \text{ --- } 2 \text{ --- } 3$ $\qquad q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3$

And so on (other examples: a single loop, a four-legged star).

Quadratic forms have been studied since at least the 1700s, most notably by Legendre and Gauss. One of the very first questions one asks when handed a given quadratic form is: Is it *positive definite*? That is, do we have $q(x_1, \ldots, x_n) \geq 0$ for all $x_1, \ldots, x_n$, and equality only for the zero vector?

Examples.

(1)

$$1 \text{ —— } 2$$ (yes)

$$q(x_1, x_2) = (x_1 - x_2/2)^2 + \frac{3}{4}x_2^2$$

(2)

$$1 \equiv 2$$ (no, *semi*definite)

$$q(x_1, x_2) = (x_1 - x_2)^2$$

(3)

$$1 \text{ ≡≡ } 2$$ (no, indefinite)

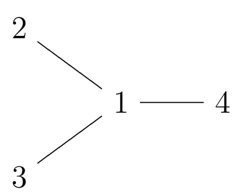
$$q(1, 1) = -1, \qquad q(1, -1) = 4$$

(4)

$$1 \text{ —— } 2 \text{ —— } 3$$ (yes, but tricky)

$$q(x_1, x_2, x_3) = (x_1 - \frac{x_2}{2})^2 + (\frac{x_2}{2} - x^3)^2 + \frac{x_2^2}{2}$$

(5)



(yes, but trickier)

$$q(x_1, x_2, x_3, x_4) = \frac{1}{4}(x_1 - 2x_2)^2 + \frac{1}{4}(x_1 - 2x_3)^2 + \frac{1}{4}(x_1 - x_4)^2 + \frac{1}{2}x_4^2$$

In general, it can be quite hard to tell immediately whether a given quadratic form is positive definite.

Easy Fact. A quadratic form q is positive semidefinite if and only if q can be written as a sum of squares of linear polynomials with positive coefficients.

Given a quadratic form that's known to be positive semidefinite, it's a linear algebra problem to decide whether it's positive definite. Still, checking whether q can be written as a sum of squares is nontrivial, and the linear algebra problem can be hard as well.

Note. Assume now that G has no loops. Then we can write q as a *matrix product*

$$q(x_1, \ldots, x_n) = \frac{1}{2} \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} & & \\ & C & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

where C is the *Cartan matrix* of G, the symmetric matrix with 2 on the diagonal and $-r$ in the (i, j) spot if there are r edges between i and j.

Example. Consider this graph again.

$$1 \text{ --- } 2 \text{ --- } 3 \tag{1}$$

The Cartan matrix is

$$C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Theorem (The Principal Minors Theorem)**.** *The quadratic form q_G is positive definite (resp., semidefinite) if all* leading principal minors *of C are positive, resp., nonnegative.*

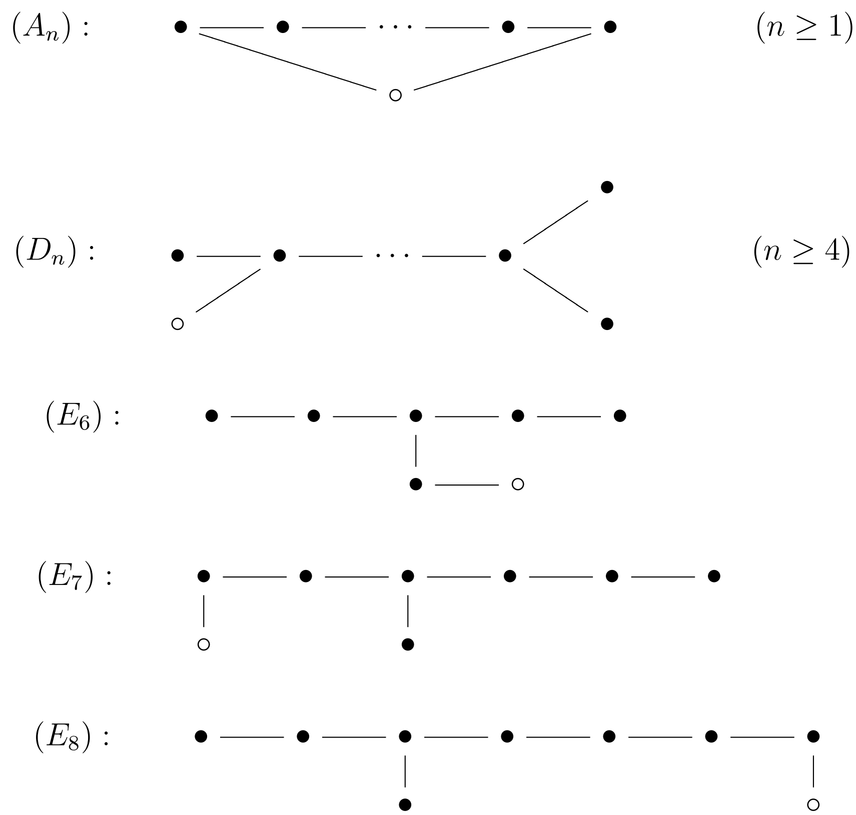
Example. The above Cartan matrix has all positive leading principal minors.

We also have the following fairly easy Lemma:

Lemma. If q_G is positive definite, thenm G has no loops, multiple edges, or vertices of degree 4 or more.

which together with the PMT gives

Theorem. *The connected graphs G for which q_G is positive definite are precisely the* Coxeter-Dynkin diagrams*:*



This theorem is essentially due to Dynkin, though he certainly didn't phrase it in this way.

(The white circles are extra vertices that don't belong in the Dynkin diagrams. Adding them in yields the *extended Dynkin diagrams*, denoted with tildes over the letters A, D, and E. The extended diagrams are precisely those graphs for which q_G is *semi*definite.)

Let's shift gears.

2. Quiver Representations

Definition. A *quiver* is just a directed graph. We again allow loops and multiple edges.

Since the aims and motivations of quiver theory are completely different from graph theory, they hardly ever get called "directed graphs" or "digraphs". It's one of my favorite mathematical terms, though.

Examples. Just add arrowheads to any of our previous examples.

Definition. A *representation of a quiver* is the following data:

- for each vertex i a finite-dimensional vector space $V_i = \mathbb{C}^n$

- for each arrow $i \longrightarrow j$ a linear transformation $T_i : V_i \longrightarrow V_j$.

Example. Note that we make no assumptions that the maps commute or anything, so we can just make something up. (And we do.)

For a fixed quiver Q, we define the *direct sum* of two representations:

$$\mathbb{C}^a \xrightarrow{A} \mathbb{C}^{a'} \qquad \oplus \qquad \mathbb{C}^b \xrightarrow{B} \mathbb{C}^{b'} \qquad = \qquad \mathbb{C}^{a+b} \xrightarrow{\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]} \mathbb{C}^{a'+b'}$$

and we have an obvious notion of *indecomposability* of a representation.

Fact. Every representation is isomorphic to a unique direct sum of indecomposable representations.

Examples.

(1)
$$\bullet$$
has only one indecomposable representation: $\mathbb{C}$. All representations are just $\mathbb{C}^n$.

(2)
$$\bullet \longrightarrow \bullet$$
Every linear transformation can be *diagonalized*, so we may choose bases to put the single matrix in the form $\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$. The indecomposables are therefore $0 \longrightarrow \mathbb{C}$, $\mathbb{C} \longrightarrow 0$, and $\mathbb{C} \xrightarrow{1} \mathbb{C}$.

(3)
$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$
The same arguments apply, but now there are six indecomposables.

(4)
$$\bullet \rightleftarrows \bullet$$
Now we can only diagonalize *one* of the matrices. There are indecomposable representations of the form $\bullet \underset{\lambda}{\overset{1}{\rightleftarrows}} \bullet$ for every $\lambda \in \mathbb{C}$, and different λ give nonisomorphic representations. There is, however, a pretty comprehensive structure theory for the representations of this quiver (mostly coming from the Jordan Normal Form). It's called the *Kronecker quiver*, and the structure theory was originally worked out by Kronecker and Weierstrass (in different terminology, of course).

(5)
$$\bullet \circlearrowright$$
For any n, $\mathbb{C}^n \circlearrowright T_n$ is indecomposable, where T_n has 1s on the superdiagonal and 0 everywhere else. (Jordan Normal Form)

As we saw in the examples, some quivers have only finitely many indecomposable representations. Which ones?

Gabriel's Theorem. A quiver has *finite representation type* if and only if its undirected graph is (a disjoint union of) Coxeter-Dynkin diagrams.

Again, the extended quivers are the answers to a slightly more permissive question. I haven't defined what *tame* representation type means, but roughly speaking it means that there is a structure theory, like the ones for the Kronecker quiver and the loop.

Nazarova's Theorem. The quivers of tame representation type are the extended Coxeter-Dynkin diagrams.

3. Other Manifestations

Another elementary one: (sub)additive functions on graphs. A function f from a graph G to the natural numbers is called *additive* if $f(n) = \frac{1}{2}\sum_{i--n} f(i)$ for every vertex n. It's *sub*additive if $\geq$ holds.

Fact. A connected graph admits a subadditive function that is not additive if and only if it is a Coxeter-Dynkin diagram. It admits an additive function if and only if it is an extended C-D diagram.

What Dynkin diagrams "really" describe are "root systems." A root system in a vector space V with inner product $(-,-)$ is a finite set of nonzero vectors $\mathcal{R}$ that spans V, and such that for every $u \in \mathcal{R}$, the so-called *orthogonal reflection*

$$v \mapsto v - 2\frac{(u,v)}{(u,u)}u$$

takes elements of $\mathcal{R}$ back into $\mathcal{R}$.

- Artinian rings with only finitely many indecomposable modules
- "simple singularities" (those having only finite many inequivalent deformations
- finite groups of symmetries generated by reflectsions (Coxeter groups, see the next talk)
- simple Lie algebras (the original area where they were introduced by Dynkin [he was still a student at Moscow State, now 82 and at Cornell])
- the Platonic Solids (!)

And there are many more. After the talk, one of my colleagues pointed out that the Cartan matrix for the A_n Dynkin diagram arises in approximation theory, and means something like "finite energy" in that context. Pretty exciting! I hope to figure out what it means someday soon.