Math. Z.
DOI 10.1007/s00209-008-0382-x

**Mathematische Zeitschrift**

ERRATUM

# On the growth of the Betti sequence of the canonical module

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**Erratum to: Math. Z. (2007) 256(3):647–659**
**DOI 10.1007/s00209-006-0096-x**

In our paper [1], three results appearing in Sect. 2 are incorrect as stated. The second part of Lemma 2.1, which assumes that $\operatorname{Ext}^i_R(M, N^\vee) = 0$ for all $i$ in a certain range and concludes an inequality on the Betti numbers of $N$, is not true for the stated range of Ext-vanishing. The correction forces changes in the statements of Theorems 2.2 and 2.4 as well.

To correct the Lemma, we adjust the statement by moving the range of indices for which $\operatorname{Ext}_R(M, N^\vee)$ is assumed to vanish, adding the codimension of $M$ to both ends of the range. This preserves the statement for maximal Cohen–Macaulay modules, where our proof did in fact apply, and makes it true for Cohen–Macaulay modules of positive codimension. (We also reword the hypotheses slightly to try to avoid possible confusion).

**Lemma 2.1** (corrected) *Let $R$ be a CM local ring, $M$ a CM $R$-module of dimension $d$, and $N$ a MCM $R$-module. Let $n$ be an integer and assume that either*

(1) $n \geq d + 1$ *and* $\operatorname{Tor}^R_i(M, N) = 0$ *for all $i$ with $n - d \leq i \leq n$, or*
(2) $n \geq 1$ *and* $\operatorname{Ext}^i_R(M, N^\vee) = 0$ *for all $i$ with $n + \dim R - d \leq i \leq n + \dim R$.*

*Then for any sequence $\boldsymbol{x} = x_1, \ldots, x_d$ regular on both $M$ and $R$,*

$$b_n(N) \leq \frac{\lambda(\mathfrak{m}M/\boldsymbol{x}M)}{\mu(M)}\, b_{n-1}(N)\,.$$

*Moreover, equality holds if and only if both $\mathfrak{m}(M/\boldsymbol{x}M \otimes_R N) = 0$ and $\mathfrak{m}(\mathfrak{m}M/\boldsymbol{x}M) = 0$.*

The online version of the original article can be found under doi:10.1007/s00209-006-0096-x.

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The only change in the statement is in case (2), which in the original reads "for all $i$ with $1 \le n \le i \le n + d$". Notice that if one were to add the additional assumption that $M$ were maximal Cohen–Macaulay, i.e. that $d = \dim R$, the two statements would coincide. The proof we gave (reproduced below) is correct in this case.

*Proof* We prove only (2).

Assume first that $M$ is MCM, so that $d = \dim R$, and induct on $d$. When $d = 0$, Matlis duality yields $\operatorname{Tor}_n^R(M, N) \cong \operatorname{Ext}_R^n(M, N^\vee)^\vee = 0$, and we get the inequality by case (1). When $d > 0$, let overlines indicate reduction modulo $x_d$ and consider the sequence $0 \longrightarrow N^\vee \xrightarrow{x_d} N^\vee \longrightarrow \overline{N^\vee} \longrightarrow 0$, which is exact as $N^\vee$ is MCM. Using the fact that $\operatorname{Hom}_{\overline{R}}(\overline{N}, \overline{\omega}) \cong \operatorname{Hom}_R(N, \omega) \otimes \overline{R}$, and the long exact sequence of Ext, we find that $\operatorname{Ext}_R^i(M, \overline{N}^\vee) = 0$ for $n \le i \le n + (d-1)$. Since $\operatorname{Ext}_R^i(M, \overline{N}^\vee) \cong \operatorname{Ext}_{\overline{R}}^i(\overline{M}, \overline{N}^\vee)$ for all $i$, and $\overline{N}^\vee$ is MCM of dimension $d - 1$ over $\overline{R}$, the induction hypothesis gives

$$b_n^{\overline{R}}(\overline{N}) \le \frac{\lambda(\overline{\mathfrak{m}}\overline{M}/\overline{\boldsymbol{x}}\overline{M})}{\mu(\overline{M})}\, b_{n-1}^{\overline{R}}(\overline{N}),$$

which is equivalent to the inequality claimed.

Now assume that $M$ has codimension $c = \dim R - d \ge 1$. Let $\boldsymbol{y} = y_1, \ldots, y_c$ be a maximal $R$- and $N^\vee$-regular sequence in the annihilator of $M$, and now let overlines indicate reduction modulo $\boldsymbol{y}$. Then $\operatorname{Ext}_R^i(M, N^\vee) \cong \operatorname{Ext}_{\overline{R}}^{i-c}(M, \overline{N^\vee})$ for all $i \ge c$, so we see that $\operatorname{Ext}_{\overline{R}}^i(M, \overline{N}^\vee) = 0$ for $n \le i \le n + \dim R - c = n + \dim \overline{R}$. As $M$ is MCM over $\overline{R}$, the previous case applies to give the inequality over $\overline{R}$, which again is equivalent to the inequality over $R$.

The statement about equality is unaffected by these changes. □

We must also shift the range of vanishing Exts assumed in Theorem 2.2. The original text had $1 \le i \le d + \mu(\omega)$ in condition (2), but the corrected Lemma 2.1 leads to the following statement.

**Theorem 2.2** (corrected) *Let $(R, \mathfrak{m})$ be a CM local ring with canonical module $\omega$, and $M$ be a CM $R$-module of dimension $d$ such that for some sequence $\boldsymbol{x}$ of length $d$ regular on both $M$ and $R$,*

(1) $\lambda(\mathfrak{m}M/\boldsymbol{x}M) < \mu(M)$, *and*
(2) $\operatorname{Ext}_R^i(M, R) = 0$ *for* $1 + \dim R - d \le i \le \dim R + \mu(\omega)$,

*then $R$ is Gorenstein. The same statement, except allowing equality in* (1)*, holds if either* $\mathfrak{m}((M/\boldsymbol{x}M) \otimes_R \omega) \ne 0$ *or* $\mathfrak{m}(\mathfrak{m}M/\boldsymbol{x}M) \ne 0$.

Similarly, the statement of Theorem 2.4 must be corrected, replacing "$\operatorname{Ext}_R^i(M, R) = 0$ for $1 \le i \le d + 1$" in condition (2) by the following.

**Theorem 2.4** (corrected) *Let $R$ be a generically Gorenstein CM local ring with canonical module $\omega$, and $M$ be a CM $R$-module of dimension $d$ such that for some sequence $\boldsymbol{x}$ of length $d$ regular on both $M$ and $R$,*

(1) $\lambda(\mathfrak{m}M/\boldsymbol{x}M) < \mu(M)$, *and*
(2) $\operatorname{Ext}_R^i(M, R) = 0$ *for* $1 + \dim R - d \le i \le \dim R + 1$,

*then $R$ is Gorenstein. The same statement, except allowing equality in (1), holds if either* $\mathfrak{m}((M/\boldsymbol{x}M) \otimes_R \omega) \neq 0$ *or* $\mathfrak{m}(\mathfrak{m}M/\boldsymbol{x}M) \neq 0$.

The derivation of both Theorems from the Lemma is as in the original.

## Reference

1. Jorgensen, D.A., Leuschke, G.J.: On the growth of the Betti sequence of the canonical module. Math. Z. **256**(3), 647–659 (2007). doi:10.1007/s00209-006-0096-x