NON-COMMUTATIVE DESINGULARIZATION OF DETERMINANTAL VARIETIES II

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ABSTRACT. In our paper "Non-commutative desingularization of determinantal varieties I" we constructed and studied non-commutative resolutions of determinantal varieties defined by maximal minors. At the end of the introduction we asserted that the results could be generalized to determinantal varieties defined by non-maximal minors, at least in characteristic zero. In this paper we prove the *existence* of non-commutative resolutions in the general case in a manner which is still characteristic free. The explicit description of the resolution by generators and relations is deferred to a later paper. As an application of our results we prove that there is a fully faithful embedding between the bounded derived categories of the two canonical (commutative) resolutions of a determinantal variety, confirming a well-known conjecture of Bondal and Orlov in this special case.

1. INTRODUCTION

Let K be a field and let F, G be two K-vector spaces of ranks m and n respectively. We take unadorned tensor products over K and denote by $(-)^\vee$ the K–dual. Put $H = \operatorname{Hom}_K(G,F)$, viewed as the affine variety of K-rational points of $\operatorname{Spec} S$, where $S = \operatorname{Sym}_K(H^\vee)$ is isomorphic to a polynomial ring in mn indeterminates. The *generic S-linear map* $\varphi\colon G\otimes S \longrightarrow F\otimes S$ corresponds to multiplication by the generic $(m\times n)$-matrix comprising those indeterminates.

Fix a non-negative integer $l < \min(m,n)$, and let $\operatorname{Spec} R$ be the locus in $\operatorname{Spec} S$ where $\bigwedge^{l+1}\varphi = 0$. Then R is the quotient of S by the ideal of $(l+1)$-minors of the generic $(m\times n)$-matrix. It is a classical result that R is Cohen-Macaulay of codimension $(n-l)(m-l)$, with singular locus defined by the l-minors of the generic matrix; in particular R is smooth in codimension 2.

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In this paper we consider some natural R-modules. For a partition $\alpha = (\alpha_1, \cdots, \alpha_r)$ and a vector space V, write

$$\bigwedge^{\alpha} V = \bigwedge^{\alpha_1} V \otimes \cdots \otimes \bigwedge^{\alpha_r} V .$$

Let α' denote the conjugate partition of α, and $\bigwedge^{\alpha'} \varphi^{\vee} \colon \bigwedge^{\alpha'} F^{\vee} \otimes S \longrightarrow \bigwedge^{\alpha'} G^{\vee} \otimes S$ the natural map induced by φ. Define

$$T_\alpha = \operatorname{image}\left(\bigwedge^{\alpha'} F^{\vee} \otimes R \xrightarrow{\left(\bigwedge^{\alpha'} \varphi^{\vee}\right) \otimes R} \bigwedge^{\alpha'} G^{\vee} \otimes R \right).$$

Our first main result generalizes [3, Theorem A], and shows that general determinantal varieties admit a *non-commutative desingularization* in the following sense. Let $B_{u,v}$ be the set of all partitions with at most u rows and at most v columns and set

$$T = \bigoplus_{\alpha \in B_{l,m-l}} T_\alpha \qquad \text{and} \qquad E = \operatorname{End}_R(T)^{\circ} .$$

Theorem A. *For $m \leqslant n$, the endomorphism ring $E = \operatorname{End}_R(T)^{\circ}$ is maximal Cohen-Macaulay as an R-module, and has moreover finite global dimension.*

In particular T_α is a maximal Cohen-Macaulay R-module for each $\alpha \in B_{l,m-l}$.

If $m = n$ then R is Gorenstein; in this case E is an example of a *non-commutative crepant resolution* as defined in [12].

The R-module T_α is in general far from indecomposable. Denote by $L_\alpha V$ the irreducible $\mathrm{GL}(V)$-module corresponding to a partition α (Schur module [14]), and assume for a moment that K has characteristic zero. Then it follows from Pieri's formula that $\bigwedge^{\alpha'} V$ is a direct sum of suitable $L_\beta V$ for $\beta \leqslant \alpha$ with $L_\alpha V$ appearing with multiplicity one. Hence if we put

$$N_\alpha = \operatorname{image}\left(L_\alpha(F^{\vee}) \otimes R \xrightarrow{(L_\alpha(\varphi^{\vee})) \otimes R} L_\alpha(G^{\vee}) \otimes R \right)$$

then in characteristic zero T_α is a direct sum of N_β for $\beta \leqslant \alpha$ with N_α appearing with multiplicity one. In particular we obtain that N_α is maximal Cohen-Macaulay. This is false in small characteristic; see Remark 4.7 below where we make the connection with Weyman's work [14, §6].

If we set $N = \bigoplus_{\alpha \in B_{l,m-l}} N_\alpha$, then $\operatorname{End}_R(N)^{\circ}$ is Morita equivalent to $\operatorname{End}_R(T)^{\circ}$. Clearly Theorem A remains valid in characteristic zero if we replace T by N.

Now let K be general again. We have taken care to state Theorem A in algebraic language but as in [3] we are only able to prove these results by invoking algebraic geometry, i.e. by constructing a suitable tilting bundle on the Springer resolution of $\operatorname{Spec} R$.

Write $\mathbb{G} = \operatorname{Grass}(l, F) \cong \operatorname{Grass}(l, m)$ for the Grassmannian variety of l-dimensional subspaces of F, and let $\pi : \mathbb{G} \longrightarrow K$ be the structure morphism to the base scheme $\operatorname{Spec} K$. On $\mathbb{G}$ we have a tautological exact sequence of vector bundles

$$0 \longrightarrow \mathcal{R} \longrightarrow \pi^* F^\vee \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{1.1.1}$$

whose fiber above a point $(V \subset F) \in \mathbb{G}$ is the short exact sequence $0 \longrightarrow (F/V)^\vee \longrightarrow F^\vee \longrightarrow V^\vee \longrightarrow 0$. We first prove the following extension of a result due to Kapranov in characteristic zero [10].

Theorem B. *The $\mathcal{O}_{\mathbb{G}}$-module*

$$\mathcal{T}_0 = \bigoplus_{\alpha \in B_{l,m-l}} \bigwedge^{\alpha'} \mathcal{Q}$$

is a classical tilting bundle on $\mathbb{G}$, i.e.

(i) $\mathcal{T}_0$ *classically generates the derived category* $\mathcal{D}^b(\operatorname{coh} \mathbb{G})$*, in that the smallest thick subcategory of* $\mathcal{D}^b(\operatorname{coh} \mathbb{G})$ *containing* $\mathcal{T}_0$ *is* $\mathcal{D}^b(\operatorname{coh} \mathbb{G})$*, and*

(ii) $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{coh} \mathbb{G})}(\mathcal{T}_0, \mathcal{T}_0[i]) = 0$ *for* $i \neq 0$*.*

From this we derive our main geometric result. Set $\mathcal{Y} = \mathbb{G} \times_{\operatorname{Spec} K} H$, with the canonical projections $p : \mathcal{Y} \longrightarrow \mathbb{G}$ and $q : \mathcal{Y} \longrightarrow H$. Define the *incidence variety*

$$\mathcal{Z} = \left\{ (V, \theta) \in \mathbb{G} \times_{\operatorname{Spec} K} H \,\middle|\, \operatorname{image} \theta \subset V \right\} \subseteq \mathcal{Y}$$

and denote by j the natural inclusion $\mathcal{Z} \longrightarrow \mathcal{Y}$. The composition $q' = qj : \mathcal{Z} \longrightarrow H$ is then a birational isomorphism from $\mathcal{Z}$ onto its image $q'(\mathcal{Z}) = \operatorname{Spec} R$, while $p' = pj : \mathcal{Z} \longrightarrow \mathbb{G}$ is a vector bundle (with zero section $\theta = 0$). Figure 1.1 summarizes the schemes and maps we have defined. We call $\mathcal{Z}$ the *Springer resolution* of $\operatorname{Spec} R$.

Theorem C. *The $\mathcal{O}_{\mathcal{Z}}$-module*

$$\mathcal{T} = p'^* \left(\bigoplus_{\alpha \in B_{l,m-l}} \bigwedge^{\alpha'} \mathcal{Q} \right)$$

is a classical tilting bundle on $\mathcal{Z}$, and furthermore

(i) $T \cong \mathbf{R} q'_* \mathcal{T}$*, and*

$$\begin{tikzcd}
\mathcal{Z} \arrow[dr, hook, "j"] \arrow[drr, bend left, "p'"] \arrow[dd, "q'"'] & & & \\
& \mathcal{Y} = H \times \mathbb{G} \arrow[r, "p"] \arrow[d, "q"'] & \mathbb{G} = \mathrm{Grass}(l, F) \arrow[r, "\pi"] & \mathrm{Spec}\, K \\
\mathrm{Spec}\, R \arrow[r, hook] & H = \mathrm{Hom}_K(G, F) & &
\end{tikzcd}$$

FIGURE 1.1.

(ii) $E \cong \mathrm{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T})^{\circ}$.

The proofs of Theorems A and C are substantially simpler than the corresponding ones in [3], even in the case of maximal minors.

As $H = \mathrm{Hom}_K(G, F)$ is canonically isomorphic to $\mathrm{Hom}_K(F^{\vee}, G^{\vee})$ we obtain a second Springer resolution map $q_2' \colon \mathcal{Z}_2 \longrightarrow \mathrm{Spec}\, R$ by replacing (F, G) with $(G^{\vee}, F^{\vee})$. As an application of Theorem C, we prove the following result.

Theorem D. *Put $\widehat{\mathcal{Z}} = \mathcal{Z} \times_H \mathcal{Z}_2$. If $m \leqslant n$ then the Fourier-Mukai transform with kernel $\mathcal{O}_{\widehat{\mathcal{Z}}}$ induces a fully faithful embedding $\mathcal{D}^b(\mathrm{coh}\, \mathcal{Z}) \hookrightarrow \mathcal{D}^b(\mathrm{coh}\, \mathcal{Z}_2)$.*

A general conjecture by Bondal and Orlov [2] asserts that a flip between algebraic varieties induces a fully faithful embedding between their derived categories. It is not hard to see that the birational map $\mathcal{Z}_2 \longrightarrow \mathcal{Z}$ is a flip, so we obtain a confirmation of the Bondal-Orlov conjecture in this special case.

In characteristic zero, we know how to describe explicitly the non-commutative desingularization as a quiver algebra with relations, as in our earlier paper [3]. This is deferred to a later paper as we want to keep the current one characteristic-free.

Characteristic-freeness complicates the representation theory somewhat, so we include a short section on the preliminaries we require, including Kempf's vanishing result and the characteristic-free versions of the Cauchy formula and Littlewood-Richardson rule. These are used to prove Theorem B in the third section. Section 4 proves Theorems A and C, and the last section contains the proof of Theorem D.

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2. Preliminaries on algebraic groups

Throughout we use [8] as a convenient reference for facts about algebraic groups. If $H \subseteq G$ is an inclusion of algebraic groups over the ground field K, then the restriction functor from rational G-modules to rational H-modules has a right adjoint denoted by ind_H^G ([8, I.3.3]). Its right derived functors are denoted by $\mathbf{R}^i \operatorname{ind}_H^G$. For an inclusion of groups $K \subseteq H \subseteq G$ and M a rational K-representation there is a spectral sequence [8, I.4.5(c)]

$$E_2^{pq} : \mathbf{R}^p \operatorname{ind}_H^G \mathbf{R}^q \operatorname{ind}_K^H M \Longrightarrow \mathbf{R}^{p+q} \operatorname{ind}_K^G M. \tag{2.0.1}$$

If G/H is a scheme and V is a finite-dimensional representation of H then $\mathcal{L}_{G/H}(V)$ is by definition the G-equivariant vector bundle on G/H given by the sections of $(G \times V)/H$. The functor $\mathcal{L}_{G/H}(-)$ defines an equivalence between the finite-dimensional H-representations and the G-equivariant vector bundles on G/H. The inverse of this functor is given by taking the fiber in $[H]$.

If G/H is a scheme then $\mathbf{R}^i \operatorname{ind}_H^G$ may be computed as [8, Prop. I.5.12]

$$\mathbf{R}^i \operatorname{ind}_H^G M = H^i(G/H, \mathcal{L}_{G/H}(M)). \tag{2.0.2}$$

We now assume that G is a split reductive group with a given split maximal torus and corresponding Borel subgroup, $T \subseteq B \subseteq G$. We let $X(T)$ be the character group of T and we identify the elements of $X(T)$ with the one-dimensional representations of T. The set of roots (the weights of $\operatorname{Lie} G$) is denoted by Δ. We have $\Delta = \Delta^- \coprod \Delta^+$ where the negative roots Δ^- represent the roots of $\operatorname{Lie} B$. For $\rho \in \Delta$ we denote the corresponding coroot in $Y(T) = \operatorname{Hom}(X(T), \mathbb{Z})$ [8, II.1.3] by $\rho^\vee$. The natural pairing between $X(T)$ and $Y(T)$ is denoted by $\langle -,- \rangle$. A weight $\alpha \in X(T)$ is dominant if $\langle \alpha, \rho^\vee \rangle \geqslant 0$ for all positive roots ρ. The set of dominant weights is denoted by $X(T)_+$, and for a dominant weight α, let $\mathcal{L}_{G/B}(\alpha)$ denote the corresponding vector bundle on G/B. We define $\operatorname{ind}_B^G \alpha$ similarly.

The following is the celebrated Kempf vanishing result ([11], see also [8, II.4.5]).

Theorem 2.1. *If $\alpha \in X(T)_+$ then $\mathbf{R}^i \operatorname{ind}_B^G \alpha = H^i(G/B, \mathcal{L}_{G/B}(\alpha))$ vanishes for $i > 0$.*

We will need the following characteristic-free version of the Cauchy formula and the Littlewood-Richardson rule. See [14, 2.3.2, 2.3.4].

Theorem 2.2 (Boffi [1], Doubilet-Rota-Stein [5]). *Let V and W be K-vector spaces and let α and β be partitions.*

(i) There is a natural filtration on $\mathrm{Sym}_t(V \otimes W)$ *whose associated graded object is a direct sum with summands tensor products $L_\gamma V \otimes L_\delta W$ of Schur functors.*

(ii) There is a natural filtration on $L_\alpha V \otimes L_\beta V$ whose associated graded object is a direct sum of Schur functors $L_\gamma V$. The γ that appear, and their multiplicities, can be computed using the usual Littlewood-Richardson rule.

In a filtration as in (ii) above, we may assume by [8, II.4.16, Remark (4)] that the $L_\gamma V$ which appear are in decreasing order for the lexicographic ordering on partitions, that is, the largest γ appears on top.

3. A TILTING BUNDLE FOR GRASSMANNIANS

In this section we prove Theorem B, the existence of a characteristic-free tilting bundle on the Grassmannian $\mathbb{G}$. We freely use the notations established in the previous sections. The proof depends on the following vanishing result which we will also use later on.

Proposition 3.1. *Let $\alpha \in B_{l,m-l}$ and let δ be any partition. Then for all $i > 0$ one has*

$$H^i\left(\mathbb{G}, \left(\bigwedge^{\alpha'} \mathcal{Q}\right)^\vee \otimes_{\mathcal{O}_\mathbb{G}} L_\delta \mathcal{Q}\right) = 0\,.$$

Before beginning the proof we introduce some more notation. We will identify $\mathbb{G} = \mathrm{Grass}(l, F)$ with $\mathrm{Grass}(m-l, F^\vee)$ via the isomorphism $(V \subset F) \mapsto ((F/V)^\vee \subset F^\vee)$.

For convenience we choose a basis $(f_i)_{i=1,\ldots,m}$ for F and a corresponding dual basis $(f_i^*)_i$ for $F^\vee$. We view $\mathbb{G}$ as the homogeneous space G/P with $G = \mathrm{GL}(m)$ and $P \subset G$ the parabolic subgroup stabilizing the point $(W \subset F^\vee) \in \mathbb{G}$, where W is spanned by $f_{l+1}^*, \ldots, f_m^*$. We let T and B be respectively the diagonal matrices and the lower triangular matrices in G. We identify $X(T)$ and $Y(T)$ with $\mathbb{Z}^m$, denoting by ε_i the i^{th} standard basis element. Thus $\sum_i a_i \varepsilon_i$ corresponds to the character $\mathrm{diag}(z_1, \ldots, z_m) \mapsto z_1^{a_1} \cdots z_m^{a_m}$. Under this identification roots and coroots coincide and are given by $\varepsilon_i - \varepsilon_j$, $i \neq j$, a root being positive if $i < j$. The pairing between $X(T)$ and $Y(T)$ is the standard Euclidean scalar product and hence $X(T)_+ = \{\sum_i a_i\varepsilon_i \mid a_i \geqslant a_j \text{ for } i \leqslant j\}$.

Let $H = G_1 \times G_2 = \mathrm{GL}(l) \times \mathrm{GL}(m-l) \subset \mathrm{GL}(m)$ be the Levi-subgroup of P containing T. We put $B_i = B \cap G_i$, $T_i = T \cap G_i$.

We fix another parabolic subgroup P° in G, given by the stabilizer of the flag spanned by $f_p^*, \dots, f_m^*$ for $p = 1, \dots, l$. We let $G^\circ = \mathrm{GL}(m-l+1) \subset P^\circ \subset G = \mathrm{GL}(m)$ be the lower right $(m-l+1 \times m-l+1)$-block in $\mathrm{GL}(m)$. We put $T^\circ = T \cap G^\circ$, $B^\circ = B \cap G^\circ$, i.e. B° is the set of lower triangular matrices in G° and T° is the set of diagonal matrices.

We also recall the following result. Cf. [6, §4, §4.8], [14, (4.1.10)].

Proposition 3.2. *Let $\delta = (\delta_1, \dots, \delta_m)$ be a partition and let $\tilde{\delta} = \sum_i \delta_i \varepsilon_i$ be the corresponding weight. Then*

$$L_\delta(F^\vee) = \mathrm{ind}_B^G \tilde{\delta}.$$

□

Proof of Proposition 3.1. Using the identity

$$\left(\wedge^a \mathcal{Q}\right)^\vee = \wedge^{l-a} \mathcal{Q} \otimes \left(\wedge^l \mathcal{Q}\right)^\vee$$

and Theorem 2.2(ii) we reduce immediately to the case $\alpha_1' = \cdots = \alpha_{m-l}' = l$. The tautological exact sequence (1.1.1) lets us write

$$\left(\wedge^l \mathcal{Q}\right)^\vee = \wedge^m F \otimes \wedge^{m-l} \mathcal{R}.$$

Thus we need to prove that

$$L_\delta \mathcal{Q} \otimes \wedge^{(m-l,\dots,m-l)} \mathcal{R}$$

(with $m-l$ instances of "$m-l$") has vanishing higher cohomology. Using (2.0.2) we see that we must prove that for $i > 0$ we have

$$\mathbf{R}^i \,\mathrm{ind}_P^G \left(L_\delta \mathcal{Q}_x \otimes \wedge^{(m-l,\dots,m-l)} \mathcal{R}_x\right) = 0, \tag{3.2.1}$$

where $x = [P] \in G/P = \mathbb{G}$. Since $\mathcal{Q}$ has rank l, we may assume that δ has at most l entries. As above we write $\tilde{\delta} = \sum_{i=1}^l \delta_i \varepsilon_i \in X(T_1)$ for the corresponding weight. Let $\sigma \in X(T_2)$ be given by $(m-l)\sum_{i=l+1}^m \varepsilon_i$ and put $\bar{\delta} = \tilde{\delta} + \sigma \in X(T)$.

As $P/B \cong (G_1 \times G_2)/(B_1 \times B_2)$ we have

$$\begin{aligned} L_\delta \mathcal{Q}_x \otimes \bigwedge^{(m-l,\dots,m-l)} \mathcal{R}_x &= \operatorname{ind}_{B_1}^{G_1} \widetilde{\delta} \otimes \operatorname{ind}_{B_2}^{G_2} \sigma \\ &= \operatorname{ind}_B^P \overline{\delta}\,. \end{aligned}$$

The positive roots of G_1 are of the form $\varepsilon_i - \varepsilon_j$ with $i < j$ and $1 \leqslant i,j \leqslant l$. Similarly the positive roots of G_2 are of the form $\varepsilon_i - \varepsilon_j$ with $i<j$ and $l+1 \leqslant i,j \leqslant m-l$. It follows that $\overline{\delta}$ is dominant when viewed as a weight for T considered as a maximal torus in $H = G_1 \times G_2$. So Kempf vanishing implies that $\mathbf{R}^i \operatorname{ind}_B^P \overline{\delta} = \mathbf{R}^i \operatorname{ind}_{B_1\times B_2}^{G_1 \times G_2} \overline{\delta} = 0$ for all $i > 0$.

Thus the spectral sequence (2.0.1) degenerates and we obtain

$$\mathbf{R}^i \operatorname{ind}_P^G \left(L_\delta \mathcal{Q}_x \otimes \bigwedge^{(m-l,\dots,m-l)} \mathcal{R}_x \right) = \mathbf{R}^i \operatorname{ind}_B^G \overline{\delta}\,. \tag{3.2.2}$$

Thus if $\overline{\delta}$ is dominant (i.e. $\delta_l \geqslant m-l$) then the desired vanishing (3.2.1) follows by invoking Kempf vanishing again.

Assume then that $\overline{\delta}$ is not dominant, i.e. $0 \leqslant \delta_l < m-l$. We claim that $\mathbf{R}^i \operatorname{ind}_B^{P^\circ} \overline{\delta} = 0$ for all i. Then by the spectral sequence (2.0.1) applied to $B \subset P^\circ \subset G$ we obtain that $\mathbf{R}^i \operatorname{ind}_B^G \overline{\delta} = 0$ for all i.

To prove the claim we note that $P^\circ/B \cong G^\circ/B^\circ$ and hence $\mathbf{R}^i \operatorname{ind}_B^{P^\circ} \overline{\delta} = \mathbf{R}^i \operatorname{ind}_{B^\circ}^{G^\circ} (\overline{\delta} \mid T^\circ)$. In other words we have reduced ourselves to the case $l = 1$ (replacing m by $m-l+1$).

We therefore assume $l=1$, so that $\mathbb{G} = \mathbb{P}^{m-1}$. The partition δ consists of a single entry δ_1 and $\sigma = \sum_{i=2}^m (m-1)\varepsilon_i$. Under the assumption $\delta_1 < m-1$ we have to prove $\mathbf{R}^i \operatorname{ind}_B^G \overline{\delta} = 0$ for all i. Applying (3.2.2) in reverse this means we have to prove that

$$\mathcal{Q}^{\otimes \delta_1} \otimes \left(\bigwedge^{(m-1,\dots,m-l)} \mathcal{R} \right)$$

has vanishing cohomology on $\mathbb{P}^{m-1}$. We now observe that the tautological sequence (1.1.1) on $\mathbb{P}^{m-1}$ takes the form

$$0 \longrightarrow \Omega_{\mathbb{P}^{m-1}}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^{m-1}}^m \longrightarrow \mathcal{O}_{\mathbb{P}^{m-1}}(1) \longrightarrow 0\,,$$

so that in particular

$$\bigwedge^{m-1} \mathcal{R} = \bigwedge^{m-1} \left(\Omega_{\mathbb{P}^{m-1}}(1) \right) = \mathcal{O}_{\mathbb{P}^{m-1}}(-1)$$

and so

$$\mathcal{Q}^{\otimes \delta_1} \otimes \bigwedge^{m-l} \mathcal{R} \otimes \cdots \otimes \bigwedge^{m-l} \mathcal{R} = \mathcal{O}_{\mathbb{P}^{m-1}}(-m+1+\delta_1)\,.$$

It is standard that this line bundle has vanishing cohomology when $\delta_1 < m-1$. $\square$

Proof of Theorem B. The main thing to prove is that $\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}_0, \mathcal{T}_0) = 0$ for $i \neq 0$. It follows from the usual spectral sequence argument that $\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}_0, \mathcal{T}_0)$ is the i^{th} cohomology of $\mathcal{H}om_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}_0, \mathcal{T}_0) = \mathcal{T}_0^\vee \otimes \mathcal{T}_0$. Applying Theorem 2.2(ii) we see that it suffices to prove that $\mathcal{T}_0^\vee \otimes L_\delta \mathcal{Q}$ has vanishing higher cohomology whenever δ is a partition with at most l rows. This is the content of Proposition 3.1.

Kapranov's resolution of the diagonal argument implies that $\mathcal{T}_0$ still classically generates $\mathcal{D}^b(\mathrm{coh}(\mathbb{G}))$ [9, §4]. For this, we must show that $L_\alpha \mathcal{Q}$ for $\alpha \in B_{l,m-l}$ is in the thick subcategory $\mathcal{C}$ generated by $\mathcal{T}$. Assume this is not the case and let α be minimal for the lexicographic ordering on partitions such that $L_\alpha \mathcal{Q}$ is *not* in $\mathcal{C}$.

Let $\alpha' = (\alpha'_1, \dots, \alpha'_{m-l})$ be the dual partition and consider $\mathcal{U} = \bigwedge^{\alpha'_1} \mathcal{Q} \otimes \cdots \otimes \bigwedge^{\alpha'_{m-l}} \mathcal{Q}$. By Theorem 2.2(ii) and the comment following, $\mathcal{U}$ maps surjectively to $L_\alpha \mathcal{Q}$ and the kernel is an extension of various $L_\beta \mathcal{Q}$ with $\beta < \alpha$. (Pieri's formula, which is a special case of the Littlewood-Richardson rule, implies that $L_\alpha \mathcal{Q}$ appears with multiplicity one in $\mathcal{U}$.) By the hypotheses all such $L_\beta \mathcal{Q}$ are in $\mathcal{C}$. Since $\mathcal{U}$ is in $\mathcal{C}$ as well we obtain that $L_\alpha \mathcal{Q}$ is in $\mathcal{C}$, which is a contradiction. $\square$

Kapranov [10] shows that

$$\mathcal{T}'_0 = \bigoplus_{\alpha \in B_{l,m-l}} L_\alpha \mathcal{Q}$$

is a tilting bundle on $\mathbb{G}$ when K has characteristic zero. For fields of positive characteristic p, Kaneda [9] shows that $\mathcal{T}'_0$ remains tilting as long as $p \geqslant m-1$. However $\mathcal{T}'_0$ fails to be tilting in very small characteristics.

Example 3.3. Assume that K has characteristic 2 and put $\mathbb{G} = \mathrm{Grass}(2,4)$. Then the short exact sequence

$$0 \longrightarrow \textstyle\bigwedge^2 \mathcal{Q} \longrightarrow \mathcal{Q} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{Q} \longrightarrow \mathrm{Sym}_2 \mathcal{Q} \longrightarrow 0 \tag{3.3.1}$$

is non-split. In particular $\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{G}}}(\mathrm{Sym}_2 \mathcal{Q}, \bigwedge^2 \mathcal{Q}) \neq 0$, so that $\mathrm{Sym}_2 \mathcal{Q}$ and $\bigwedge^2 \mathcal{Q}$ are not common direct summands of a tilting bundle on $\mathbb{G}$.

To see that (3.3.1) is not split, tensor with $(\bigwedge^2 \mathcal{Q})^\vee$ to obtain the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{G}} \longrightarrow \mathcal{E}nd_{\mathcal{O}_{\mathbb{G}}}(\mathcal{Q}) \longrightarrow (\textstyle\bigwedge^2 \mathcal{Q})^\vee \otimes \mathrm{Sym}_2 \mathcal{Q} \longrightarrow 0 \tag{3.3.2}$$

where the leftmost map is the obvious one. Any splitting of the inclusion $\mathcal{O}_{\mathbb{G}} \longrightarrow \mathcal{E}nd_{\mathcal{O}_{\mathbb{G}}}(\mathcal{Q})$ is of the form $\mathrm{Tr}(a-)$, where Tr is the reduced trace and a is an element of $\mathrm{End}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{Q})$ such that $\mathrm{Tr}(a) = 1$. Hence it is sufficient to prove that $\mathrm{End}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{Q}) = K$ since in that case we have $\mathrm{Tr}(a) = 0$ for any $a \in \mathrm{End}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{Q})$.

By (the proof of) Proposition 3.1 we have $H^i(\mathbb{G}, (\bigwedge^2 \mathcal{Q})^\vee \otimes \mathrm{Sym}_2 \mathcal{Q}) = 0$ for all $i \geqslant 0$ (observe that if we go through the proof we obtain a situation where $\overline{\delta}$ is not dominant, whence all cohomology vanishes) and of course we also have $H^0(\mathbb{G}, \mathcal{O}_{\mathbb{G}}) = K$. Applying $H^0(\mathbb{G}, -)$ to (3.3.2) thus shows $\mathrm{End}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{Q}) = K$.

Remark 3.4. By [4, Lemma (3.4)] we obtain (at least when K is algebraically closed) a more economical tilting bundle for $\mathbb{G}$,

$$\widetilde{\mathcal{T}} = \bigoplus_{\alpha \in B_{l,m-l}} \mathcal{L}_{\mathbb{G}}(M(\alpha)),$$

where $M(\alpha)$ is the tilting $\mathrm{GL}(l)$-representation with highest weight α. Note however that the character of $M(\alpha)$ strongly depends on the characteristic, whence so does the nature of $\widetilde{\mathcal{T}}$.

4. A TILTING BUNDLE ON THE RESOLUTION

To prove Theorem C, keep all the notation introduced there. One easily verifies that

$$\mathcal{Z} = \underline{\mathrm{Spec}}\left(\mathrm{Sym}_{\mathbb{G}}(G \otimes \mathcal{Q})\right);$$

indeed, a closed point of the right-hand side consists of a pair $(V \subset F, \theta)$, where $(V \subset F) \in \mathbb{G}$ and θ is an element of the fiber of $(G \otimes \mathcal{Q})^\vee$ over the point $(V \subset F)$. That fiber is $(G \otimes V^\vee)^\vee = \mathrm{Hom}_K(G, V) \subset \mathrm{Hom}_K(G, F)$, so the pair (V, θ) is precisely a point of $\mathcal{Z}$.

Set $\mathcal{T} = p'^* \mathcal{T}_0$, a vector bundle on $\mathcal{Z}$.

Proposition 4.1. *The $\mathcal{O}_{\mathcal{Z}}$-module $\mathcal{T} = p'^* \mathcal{T}_0$ is a tilting bundle on $\mathcal{Z}$.*

Proof. Since $\mathcal{T}_0$ classically generates $\mathcal{D}^b(\mathrm{coh}\, \mathbb{G})$ it is easy to see that $\mathcal{T}$ classically generates $\mathcal{D}^b(\mathrm{coh}\, \mathcal{Z})$, so it remains to prove Ext-vanishing. We have

$$\mathrm{Ext}^i_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T}, \mathcal{T}) = H^i(\mathbb{G}, \mathrm{Sym}_{\mathbb{G}}(G \otimes \mathcal{Q}) \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{E}nd_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}_0))$$

and hence we need to prove that

$$\mathrm{Sym}_{\mathbb{G}}(G \otimes \mathcal{Q}) \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{H}om_{\mathcal{O}_{\mathbb{G}}}(\wedge^{\alpha'} \mathcal{Q}, \wedge^{\beta'} \mathcal{Q}) \tag{4.1.1}$$

has vanishing higher cohomology for $\alpha, \beta \in B_{l,m-l}$.

Using Theorem 2.2 we find that (4.1.1) has a filtration whose associated graded object is a direct sum of vector bundles of the form

$$(\wedge^{\alpha'} \mathcal{Q})^{\vee} \otimes_{\mathcal{O}_{\mathbb{G}}} L_{\delta} \mathcal{Q} \tag{4.1.2}$$

where $\alpha \in B_{l,m-l}$ and δ is any partition containing β. It now suffices to invoke Proposition 3.1. □

To prove the rest of Theorem C, we shall show that $\mathrm{End}_R(\mathbf{R}q'_* \mathcal{T})^{\circ} = \mathbf{R}q'_* \mathcal{E}nd_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T})^{\circ}$, and that the latter is MCM and has finite global dimension. Put

$$\mathcal{E} = \mathcal{E}nd_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T})^{\circ},$$

and let $\omega_{\mathcal{Z}}$ be the dualizing sheaf of $\mathcal{Z}$.

Lemma 4.2. *Assume $m \leqslant n$. Then $\mathrm{Ext}^i_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{E}, \omega_{\mathcal{Z}}) = 0$ for all $i > 0$.*

Proof. We have $\mathcal{E} = p'^* \mathcal{E}_0$, with $\mathcal{E}_0 = \mathcal{H}om_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}_0, \mathcal{T}_0)$. Substituting this and using the fact that $\mathcal{E}_0$ is self-dual, we find

$$\begin{aligned} \mathrm{Ext}^i_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{E}, \omega_{\mathcal{Z}}) &= \mathrm{Ext}^i_{\mathcal{O}_{\mathcal{Z}}}(p'^* \mathcal{E}_0, \omega_{\mathcal{Z}}) \\ &= \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(\mathcal{E}_0, p'_* \omega_{\mathcal{Z}}) \\ &= H^i(\mathbb{G}, \mathcal{E}_0 \otimes_{\mathcal{O}_{\mathbb{G}}} p'_* \omega_{\mathcal{Z}}). \end{aligned}$$

Hence to continue we must be able to compute $p'_* \omega_{\mathcal{Z}}$. Since $\mathcal{Z} = \underline{\mathrm{Spec}}\big(\mathrm{Sym}_{\mathbb{G}}(G \otimes \mathcal{Q})\big)$, the standard expression for the dualizing sheaf of a symmetric algebra gives

$$p'_* \omega_{\mathcal{Z}} = \omega_{\mathbb{G}} \otimes_{\mathcal{O}_{\mathcal{Z}}} \wedge^{ln}(G \otimes \mathcal{Q}) \otimes_{\mathcal{O}_{\mathcal{Z}}} \mathrm{Sym}_{\mathbb{G}}(G \otimes \mathcal{Q}).$$

Furthermore the sheaf $\Omega_{\mathbb{G}}$ of differential forms on $\mathbb{G}$ is known to be given by $\Omega_{\mathbb{G}} = \mathcal{Q}^{\vee} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{R}$, where $\mathcal{R}$ is the tautological sub-bundle of $\pi^* F^{\vee}$ as in (1.1.1). Hence $\omega_{\mathbb{G}} = \wedge^{ln}(\mathcal{Q}^{\vee} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{R})$ and so

$$p'_* \omega_{\mathcal{Z}} = \wedge^{ln}(\mathcal{Q}^{\vee} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{R}) \otimes_{\mathcal{O}_{\mathbb{G}}} \wedge^{ln}(G \otimes \mathcal{Q}) \otimes_{\mathcal{O}_{\mathbb{G}}} \mathrm{Sym}_{\mathbb{G}}(G \otimes \mathcal{Q}).$$

Rewriting all the exterior powers in terms of $\mathcal{Q}$, we find

$$\begin{aligned}
&\textstyle\bigwedge^{ln}(\mathcal{Q}^\vee \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{R}) \otimes_{\mathcal{O}_{\mathbb{G}}} \bigwedge^{ln}(G \otimes \mathcal{Q}) \\
&\qquad = \left(\textstyle\bigwedge^l \mathcal{Q}\right)^{-m+l} \otimes_{\mathcal{O}_{\mathbb{G}}} \left(\textstyle\bigwedge^{m-l}\mathcal{R}\right)^l \otimes_{\mathcal{O}_{\mathbb{G}}} (\textstyle\bigwedge^n G)^l \otimes \left(\textstyle\bigwedge^l \mathcal{Q}\right)^n \\
&\qquad = \left(\textstyle\bigwedge^l \mathcal{Q}\right)^{-m+l} \otimes_{\mathcal{O}_{\mathbb{G}}} (\textstyle\bigwedge^m F)^{-l} \otimes \left(\textstyle\bigwedge^l \mathcal{Q}\right)^{-l} \otimes_{\mathcal{O}_{\mathbb{G}}} (\textstyle\bigwedge^n G)^l \otimes_{\mathcal{O}_{\mathbb{G}}} \left(\textstyle\bigwedge^l \mathcal{Q}\right)^n \\
&\qquad = \left(\textstyle\bigwedge^l \mathcal{Q}\right)^{n-m} \otimes (\textstyle\bigwedge^m F)^{-l} \otimes (\textstyle\bigwedge^n G)^l .
\end{aligned}$$

So finally

$$\mathcal{E}_0 \otimes_{\mathcal{O}_{\mathbb{G}}} p'_* \omega_{\mathcal{Z}} = (\textstyle\bigwedge^m F)^{-l} \otimes (\textstyle\bigwedge^n G)^l \otimes \mathcal{E}_0 \otimes_{\mathcal{O}_{\mathbb{G}}} \left(\textstyle\bigwedge^l \mathcal{Q}\right)^{n-m} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathrm{Sym}_{\mathbb{G}}(G \otimes \mathcal{Q}).$$

Discarding the vector spaces $\bigwedge^m F$ and $\bigwedge^n G$, we find a direct sum of vector bundles of the form

$$\textstyle\bigwedge^{\alpha'} \mathcal{Q}^\vee \otimes_{\mathcal{O}_{\mathbb{G}}} \bigwedge^\beta \mathcal{Q} \otimes_{\mathcal{O}_{\mathbb{G}}} \left(\bigwedge^l \mathcal{Q}\right)^{n-m} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathrm{Sym}_{\mathbb{G}}(G \otimes \mathcal{Q}),$$

which (since $m \leqslant n$) are the subject of Proposition 3.1. □

Next we verify Theorem C for

$$\overline{E} = \mathrm{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T})^\circ = \Gamma(\mathcal{Z}, \mathcal{E}) \qquad \text{and} \qquad \overline{T} = \Gamma(\mathcal{Z}, \mathcal{T}).$$

Recall the following consequence of tilting (see e.g. [7]).

Proposition 4.3. *Assume that $\mathcal{T}$ is a tilting bundle on a smooth variety X. Then $\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}, -)$ defines an equivalence of derived categories $\mathcal{D}^b(\mathrm{coh}\,X) \cong \mathcal{D}^b(\mathrm{mod}\,E)$ where $E = \mathrm{End}_{\mathcal{O}_X}(\mathcal{T})^\circ$. If X is projective over an affine variety then E is finite over its center and has finite global dimension.*

Proposition 4.4. *Assume $m \leqslant n$. Then*

(i) $\overline{E} \cong \mathrm{End}_R(\overline{T})^\circ$;
(ii) $\overline{E}$ *and* $\overline{T}$ *are MCM R-modules; and*
(iii) $\overline{E}$ *has finite global dimension.*

Proof. That $\overline{E}$ has finite global dimension follows from Propositions 4.1 and 4.3. Since $\mathrm{Ext}^i_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T}, \mathcal{T}) = 0$ for $i > 0$ by Proposition 4.1, the higher direct images of $\mathcal{E}$ vanish, i.e.

$$\mathbf{R}q'_* \mathcal{E} = q'_* \mathcal{E} = \overline{E}.$$

To prove that $\overline{E}$ is MCM we must show that $\mathrm{Ext}^i_R(\overline{E},\omega_R) = 0$ for $i > 0$, where ω_R is the dualizing module for R. Replacing $\overline{E}$ by $\mathbf{R}q'_*\mathcal{E}$ and using duality for the proper morphism q' [14, 1.2.22], we see that this is equivalent to showing $\mathrm{Ext}^i_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{E}, q'^!\omega_R) = 0$ for $i > 0$. But $q'^!\omega_R = \omega_{\mathcal{Z}}$ is the dualizing sheaf for $\mathcal{Z}$, so Lemma 4.2 implies that $\overline{E}$ is MCM.

As $\mathcal{O}_{\mathcal{Z}}$ is a direct summand of $\mathcal{T}$ we see that $\overline{T}$ is a summand of $\overline{E}$, whence $\overline{T}$ is Cohen-Macaulay as well. Furthermore we have an obvious homomorphism $i\colon \mathrm{End}_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{T}) \longrightarrow \mathrm{End}_R(\overline{T})$ between reflexive R-modules, which is an isomorphism on the locus where $q'\colon \mathcal{Z} \longrightarrow \mathrm{Spec}\, R$ is an isomorphism. The complement of this locus is given by the matrices which have rank $< l$, a subvariety of $\mathrm{Spec}\, R$ of codimension $\geqslant 2$. Hence i is an isomorphism. $\square$

Propositions 4.1 and 4.4 imply Theorems A and C provided we can show $T \cong \overline{T}$. We do this next. Recall that for a partition α we denote

$$N_\alpha = \mathrm{image}\left(L_\alpha(F^\vee) \otimes R \xrightarrow{(L_\alpha(\varphi^\vee))\otimes R} L_\alpha(G^\vee) \otimes R\right).$$

Proposition 4.5. *With notation as above, we have*

$$N_\alpha \cong \Gamma(\mathcal{Z}, p'^* L_\alpha \mathcal{Q}).$$

Proof. With $\varphi\colon G \otimes S \longrightarrow F \otimes S$ the generic map defined over S, let $\psi = j^* q^* \varphi$ be the map induced over $\mathcal{Z}$. Then the fiber of $\psi^\vee$ over a point (V, θ) factors as

$$F^\vee \longrightarrow V^\vee \longrightarrow G^\vee$$

where the first map is the dual of the given inclusion $V \hookrightarrow F$. Thus we obtain that $\psi^\vee$ factors as

$$p'^* \pi^* F^\vee \longrightarrow p'^* \mathcal{Q} \longrightarrow p'^* \pi^* G^\vee.$$

The first map is obviously surjective. The second map is injective since it is a map between vector bundles which is generically injective. By exactness of the Schur functors applied to vector bundles, we get an epi-mono factorization

$$L_\alpha(\psi^\vee)\colon L_\alpha(p'^*\pi^* F^\vee) \longrightarrow L_\alpha p'^* \mathcal{Q} \longrightarrow L_\alpha(p'^* \pi^* G^\vee).$$

To prove the claim it is clearly sufficient to show that the first map remains an epimorphism after applying q'_*, i.e. that the epimorphism

$$\pi^* L_\alpha(F^\vee) \otimes_{\mathcal{O}_{\mathbb{G}}} \mathrm{Sym}_{\mathbb{G}}(G \otimes \mathcal{Q}) \longrightarrow L_\alpha \mathcal{Q} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathrm{Sym}_{\mathbb{G}}(G \otimes \mathcal{Q})$$

remains an epimorphism upon applying $\Gamma(\mathbb{G},-)$. In fact it suffices to show that

$$\pi^*\left(L_\alpha(F^\vee)\otimes_{\mathcal{O}_\mathbb{G}}\mathrm{Sym}_\mathbb{G}(G\otimes F^\vee)\right)\longrightarrow L_\alpha\mathcal{Q}\otimes_{\mathcal{O}_\mathbb{G}}\mathrm{Sym}_\mathbb{G}(G\otimes\mathcal{Q})$$

remains an epimorphism upon applying $\Gamma(\mathbb{G},-)$. By Theorem 2.2, source and target are filtered by Schur functors, so it is enough to show that for any partition δ the canonical map

$$\pi^*L_\delta(F^\vee)\longrightarrow L_\delta\mathcal{Q}$$

remains an epimorphism upon applying $\Gamma(\mathbb{G},-)$. But taking global sections of this map gives

$$L_\delta(F^\vee)\longrightarrow\Gamma(\mathbb{G},L_\delta\mathcal{Q})$$

which is even an isomorphism by the definition of Schur modules. Hence we are done. □

Set $\overline{T}_\alpha=\Gamma(\mathcal{Z},\mathcal{T}_\alpha)$, where $\mathcal{T}_\alpha=p'^*(\bigwedge^{\alpha'}\mathcal{Q})$ as in Theorem B, and recall

$$T_\alpha=\mathrm{image}\left(\bigwedge^{\alpha'}(F^\vee)\otimes R\xrightarrow{(\bigwedge^{\alpha'}\varphi^\vee)\otimes R}\bigwedge^{\alpha'}(G^\vee)\otimes R\right).$$

Filtering everything by Schur functors and applying Proposition 4.5, we see that these coincide:

Corollary 4.6. *We have $T_\alpha\cong\overline{T}_\alpha$ for each $\alpha\in B_{l,m-l}$. In particular $T\cong\overline{T}$ is a maximal Cohen-Macaulay R-module.*

Assembling the pieces, we obtain Theorem C and, as a consequence, Theorem A.

Remark 4.7. It follows from Proposition 4.5 that $N_\alpha=M(\alpha,0)$ in the notation of [14, §6]. In particular the very general result [14, Cor (6.5.17)] gives an alternative way to see that N_α is Cohen-Macaulay in characteristic zero. Furthermore [14, Example (6.5.18)] shows that N_2 is not Cohen-Macaulay in characteristic 2.

Example 4.8. Assume that $l=m-1$ with $m\leqslant n$. Then we have $\mathbb{G}=\mathbb{P}^{m-1}$. Set $\mathbb{P}=\mathbb{P}^{m-1}$, so that $\mathcal{Q}=\Omega^\vee_\mathbb{P}(-1)$, and let $\alpha=1^a$ for some a, $0\leqslant a\leqslant m-1$. We find

$$\begin{aligned}\mathcal{T}_\alpha&=p'^*\left(\textstyle\bigwedge^a\Omega^\vee_\mathbb{P}(-a)\right)\\&=p'^*\left(\textstyle\bigwedge^{m-1-a}\Omega_\mathbb{P}\otimes_{\mathcal{O}_\mathbb{P}}\omega_\mathbb{P}^{-1}(-a)\right)\\&=p'^*\left(\textstyle\bigwedge^{m-1-a}\Omega_\mathbb{P}(m-a)\right)\end{aligned}$$

Thus in the notation of [3] we have $T_\alpha=M_{m-a}$.

5. Proof of Theorem D

We now need to refer to the two resolutions of $\operatorname{Spec} R$ in a uniform way, so we introduce appropriate symmetrical notation. We start by putting $G_1 = F^\vee$ and $G_2 = G$ so that

$$H = \operatorname{Sym}_K(G_1 \otimes G_2).$$

We also put $n_i = \operatorname{rank}_K G_i$ and $\mathbb{G}_i = \operatorname{Grass}(n_i - l, G_i)$. Thus $n_1 = m$, $n_2 = n$, and we have canonically $\mathbb{G}_1 \cong \mathbb{G}$.

For symmetry we also put $\mathcal{Z}_1 = \mathcal{Z}$. In general we will decorate the notations in the diagram (1.1) by a "1" or a "2" depending on whether they refer to $\mathcal{Z}_1$ or $\mathcal{Z}_2$.

We now explain how we prove Theorem D. In Proposition 4.1 we have constructed tilting bundles $\mathcal{T}_1$, $\mathcal{T}_2$ on $\mathcal{Z}_1$, $\mathcal{Z}_2$. For our purposes it turns out to be technically more convenient to use the tilting bundle $\mathcal{T}_1^\vee$ on $\mathcal{Z}_1$ rather than $\mathcal{T}_1$. With E_1', E_2 the endomorphism rings of $\mathcal{T}_1^\vee$ and $\mathcal{T}_2$ respectively, it turns out that if $n_1 \leqslant n_2$ then $E_1' \cong eE_2e$ for a suitable idempotent $e \in E_2$. Thus we immediately obtain a fully faithful embedding $D^b(\operatorname{coh} \mathcal{Z}_1) \hookrightarrow D^b(\operatorname{coh} \mathcal{Z}_2)$. We then show that this embedding coincides with the indicated Fourier-Mukai transform.

Now we proceed with the actual proof. On $\mathbb{G}_i$ we have tautological exact sequences

$$0 \longrightarrow \mathcal{R}_i \longrightarrow \pi_i^* G_i \longrightarrow \mathcal{Q}_i \longrightarrow 0.$$

We also define

$$\widehat{\mathcal{Z}} = \mathcal{Z}_1 \times_H \mathcal{Z}_2.$$

There are projection maps $r_1 \colon \widehat{\mathcal{Z}} \longrightarrow \mathcal{Z}_1$, $r_2 \colon \widehat{\mathcal{Z}} \longrightarrow \mathcal{Z}_2$. These fit together in the following commutative diagram.

$$\begin{tikzcd}
 & & \widehat{\mathcal{Z}} \arrow[dl, "r_1"] \arrow[dr, "r_2"'] \arrow[dll, bend right, "p_1''"'] \arrow[drr, bend left, "p_2''"] & & \\
 & \mathcal{Z}_1 \arrow[dl, "p_1'"] \arrow[dr, "q_1'"] & & \mathcal{Z}_2 \arrow[dl, "q_2'"'] \arrow[dr, "p_2'"'] & \\
\mathbb{G}_1 & & \operatorname{Spec} R & & \mathbb{G}_2
\end{tikzcd}$$

Let $H_0 \subset \operatorname{Spec} R$ be the (open) locus of tensors of rank exactly l, so that the maps q_i' and r_i, for $i = 1,2$, are all isomorphisms above H_0. Let $\widehat{\mathcal{Z}}_0$ be the inverse image of H_0 in $\widehat{\mathcal{Z}}$.

Let α be a partition and set $\mathcal{T}_{\alpha,i} = p_i'^{\,*}\left(\bigwedge^{\alpha'}\mathcal{Q}_i\right)$ for $i = 1,2$. Further set $B_i = B_{l,n_i-l}$,

$$\mathcal{T}_i = \bigoplus_{\alpha\in B_i}\mathcal{T}_{\alpha,i} \qquad \text{and} \qquad E_i = \operatorname{End}_{\mathcal{O}_{\mathcal{Z}_i}}(\mathcal{T}_i)^\circ.$$

By Theorem C, $\mathcal{T}_i$ is a tilting bundle on $\mathcal{Z}_i$ and hence $\mathcal{D}^b(\operatorname{coh}\mathcal{Z}_i) \cong \mathcal{D}^b(\operatorname{mod} E_i)$.

Here is an asymmetrical piece of notation. Assume that $n_1 \leqslant n_2$. Then $B_1 \subseteq B_2$. Set

$$\mathcal{T}_2' = \bigoplus_{\alpha\in B_1}\mathcal{T}_{\alpha,2} \subset \bigoplus_{\alpha\in B_2}\mathcal{T}_{\alpha,2} = \mathcal{T}_2 \qquad \text{and} \qquad E_2' = \operatorname{End}_{\mathcal{O}_{\mathcal{Z}_2}}(\mathcal{T}_2')^\circ. \tag{5.0.1}$$

As $\mathcal{T}_2'$ is a direct summand of $\mathcal{T}_2$, we have $E_2' = eE_2e$ for a suitable idempotent $e \in E_2$. Hence there is a fully faithful embedding

$$\widetilde{e}\colon \mathcal{D}^b(\operatorname{mod} E_2') \hookrightarrow \mathcal{D}^b(\operatorname{mod} E_2) \tag{5.0.2}$$

given by $\widetilde{e}(\mathcal{M}) = E_2e \otimes_{E_2'} \mathcal{M}$.

Put $E_1' = \operatorname{End}_{\mathcal{O}_{\mathcal{Z}_1}}(\mathcal{T}_1^\vee)^\circ$. Note that it follows easily from Grothendieck duality that $\mathcal{T}_1^\vee$ is also a tilting bundle on $\mathcal{Z}_1$.

Finally set

$$T_{\alpha,i} = q'_{i*}\mathcal{T}_{\alpha,i}, \qquad T_i = q'_{i*}\mathcal{T}_i,$$

and $T_2' = q'_{2*}\mathcal{T}_2'$. By Theorem C, we have $E_i = \operatorname{End}_R(T_i)^\circ$, $E_1' = \operatorname{End}_R(T_1^\vee)^\circ$, and $E_2' = \operatorname{End}_R(T_2')^\circ$.

Lemma 5.1. *One has* $\widehat{\mathcal{Z}} = \underline{\operatorname{Spec}}\left(\operatorname{Sym}_{\mathbb{G}_1\times\mathbb{G}_2}(\mathcal{Q}_1 \boxtimes \mathcal{Q}_2)\right)$.

Proof. This is a straightforward computation.

$$\begin{aligned}
\mathcal{Z}_1 \times_H \mathcal{Z}_2 &= \mathcal{Z}_1 \times_{\mathbb{G}_1\times H} (\mathbb{G}_1 \times H) \times_H (\mathbb{G}_2 \times H) \times_{\mathbb{G}_2\times H} \mathcal{Z}_2 \\
&= \mathcal{Z}_1 \times_{\mathbb{G}_1\times H} (\mathbb{G}_1 \times \mathbb{G}_2 \times H) \times_{\mathbb{G}_2\times H} \mathcal{Z}_2 \\
&= (\mathcal{Z}_1 \times \mathbb{G}_2) \times_{\widehat{\mathbb{G}}\times H} (\mathcal{Z}_2 \times \mathbb{G}_1) \\
&= \underline{\operatorname{Spec}}\left(\operatorname{Sym}_{\mathbb{G}_1\times\mathbb{G}_2}(\mathcal{Q}_1 \boxtimes \pi_2^* G_2) \otimes_{\operatorname{Sym}_{\mathbb{G}_1\times\mathbb{G}_2}(\pi_1^*G_1 \boxtimes \pi_2^*G_2)} \operatorname{Sym}_{\mathbb{G}_1\times\mathbb{G}_2}(\pi_1^*G_1 \boxtimes \mathcal{Q}_2)\right) \\
&= \underline{\operatorname{Spec}}\operatorname{Sym}_{\mathbb{G}_1\times\mathbb{G}_2}(\mathcal{Q}_1 \boxtimes \mathcal{Q}_2)
\end{aligned}$$

□

Proposition 5.2. *Assume $n_1 \leqslant n_2$. Then $T_2' \cong T_1^\vee$. In particular $E_2' \cong E_1'$, and there is a fully faithful embedding $\mathcal{D}^b(\operatorname{mod} E_1') \hookrightarrow \mathcal{D}^b(\operatorname{mod} E_2)$ (using* (5.0.2)*).*

Proof. Since $\widehat{\mathcal{Z}} = \underline{\mathrm{Spec}}\big(\mathrm{Sym}_{\mathbb{G}_1\times\mathbb{G}_2}(\mathcal{Q}_1 \boxtimes \mathcal{Q}_2)\big)$, we have a canonical map

$$u\colon (p_2'')^*\mathcal{Q}_2 \longrightarrow (p_1'')^*\mathcal{Q}_1^\vee$$

which is an isomorphism on $\widehat{\mathcal{Z}}_0$. Apply $\bigwedge^{\alpha'}(-)$ for a partition α to obtain a map

$$\bigwedge\nolimits^{\alpha'} u\colon r_2^*\mathcal{T}_{\alpha,2} \longrightarrow r_1^*(\mathcal{T}_{\alpha,1})^\vee \tag{5.2.1}$$

and push down with $(q_1'r_1)_* = (q_2'r_2)_*$ to get a homomorphism of R-modules

$$\tau_\alpha\colon T_{\alpha,2} \longrightarrow T_{\alpha,1}^\vee \tag{5.2.2}$$

which is an isomorphism on H_0. Letting α run over partitions in B_1, we find a homomorphism $\tau\colon T_2' \longrightarrow T_1^\vee$ which is also an isomorphism on H_0. Since the exceptional loci for the q_i' in $\mathcal{Z}_i$ have codimension at least 2, the modules T_1 and T_2' are reflexive by [13, Lemma 4.2.1]. (In fact we know already that T_1 is Cohen-Macaulay.) Hence $\tau\colon T_2' \longrightarrow T_1^\vee$ is an isomorphism.

In particular τ induces an isomorphism $\widetilde{\tau}\colon E_1' \longrightarrow E_2'$. $\square$

The birational map $\mathcal{Z}_2 \longrightarrow \mathcal{Z}_1$ is easily seen to be a *flip*. Our final result thus verifies, in this special case, a general conjecture of Bondal and Orlov [2].

Theorem 5.3. *Assume $n_1 \leqslant n_2$. Then there is a fully faithful embedding*

$$\mathcal{F}\colon \mathcal{D}^b(\mathrm{coh}\,\mathcal{Z}_1) \longrightarrow \mathcal{D}^b(\mathrm{coh}\,\mathcal{Z}_2)$$

given by

$$\mathcal{F}(\mathcal{M}) = \mathcal{T}_2' \overset{\mathbf{L}}{\otimes}_{E_1'} \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}_1}}(\mathcal{T}_1^\vee, \mathcal{M})$$

where $E_1' = \mathrm{End}_R(\mathcal{T}_1^\vee)^\circ$ acts on $\mathcal{T}_2'$ via the isomorphism $E_1' \cong \mathrm{End}_{\mathcal{O}_{\mathcal{Z}_2}}(\mathcal{T}_2')^\circ$ of Proposition 5.2.

Proof. Since $\mathcal{T}_1^\vee$ and $\mathcal{T}_2$ are tilting on $\mathcal{Z}_1$ and $\mathcal{Z}_2$, respectively, we have equivalences

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}_1}}(\mathcal{T}_1^\vee, -)\colon \mathcal{D}^b(\mathrm{coh}\,\mathcal{Z}_1) \longrightarrow \mathcal{D}^b(\mathrm{mod}\,E_1')$$

and

$$\mathcal{T}_2 \overset{\mathbf{L}}{\otimes}_{E_2} -\colon \mathcal{D}^b(\mathrm{mod}\,E_2) \longrightarrow \mathcal{D}^b(\mathrm{coh}\,\mathcal{Z}_2).$$

Putting these together with the isomorphism $E_1' \cong E_2'$ and the fully faithful embedding $\widetilde{e}\colon \mathcal{D}^b(\operatorname{mod} E_2') \longrightarrow \mathcal{D}^b(\operatorname{mod} E_2)$, we find the composition

$$\mathcal{F}\colon \mathcal{D}^b(\operatorname{coh}\mathcal{Z}_1) \xrightarrow{\cong} \mathcal{D}^b(\operatorname{mod} E_1') \xrightarrow{\cong} \mathcal{D}^b(\operatorname{mod} E_2') \hookrightarrow \mathcal{D}^b(\operatorname{mod} E_2) \xrightarrow{\cong} \mathcal{D}^b(\operatorname{coh}\mathcal{Z}_2),$$

of the form asserted. □

Theorem 5.4. *Assume that $n_1 \leqslant n_2$. Then the Fourier-Mukai transform $\mathsf{FM} = \mathbf{R}r_{2*}\mathbf{L}r_1^*$ with kernel $(r_1,r_2)_*\mathcal{O}_{\widehat{\mathcal{Z}}}$ defines a fully faithful embedding*

$$\mathsf{FM}\colon \mathcal{D}^b(\operatorname{coh}\mathcal{Z}_1) \hookrightarrow \mathcal{D}^b(\operatorname{coh}\mathcal{Z}_2).$$

There is a natural isomorphism between FM and the functor $\mathcal{F} = \mathcal{T}_2' \overset{\mathbf{L}}{\otimes}_{E_1'} \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}_1}}(\mathcal{T}_1^\vee, -)$ introduced in Proposition 5.3. In particular FM is fully faithful.

Proof. For a partition $\alpha \in B_1$, the map $\bigwedge^{\alpha'} u\colon r_2^*\mathcal{T}_{\alpha,2} \longrightarrow r_1^*(\mathcal{T}_{\alpha,1})^\vee$ constructed in (5.2.1) gives by adjointness a homomorphism on $\mathcal{Z}_2$

$$\sigma\colon \mathcal{T}_{\alpha,2} \longrightarrow \mathbf{R}r_{2*}r_1^*(\mathcal{T}_{\alpha,1})^\vee.$$

We claim that σ is an isomorphism. In particular we must show $\mathbf{R}^i r_{2*}r_1^*(\mathcal{T}_{\alpha,1})^\vee = 0$ for $i > 0$. To this latter end it is sufficient to show that for all $y \in \mathbb{G}_2$ and all $i > 0$ we have

$$H^i(\mathbb{G}_1, \textstyle\bigwedge^{\alpha'} \mathcal{Q}_1^\vee \otimes_{\mathcal{O}_{\mathbb{G}_1}} \mathrm{Sym}_{\mathbb{G}_1}(\mathcal{Q}_1 \otimes (\mathcal{Q}_2)_y)) = 0.$$

This follows again from the Cauchy formula together with Proposition 3.1.

Now we can see that $\sigma\colon \mathcal{T}_{\alpha,2} \longrightarrow r_{2*}r_1^*(\mathcal{T}_{\alpha,1})^\vee$ is an isomorphism. The source is reflexive, the target is torsion-free, and over $\widehat{\mathcal{Z}}_0$ the map σ coincides with $(q_2')^*\tau_\alpha$, where $\tau_\alpha\colon T_{\alpha,2} \longrightarrow T_{\alpha,1}^\vee$ as in (5.2.2). Since each τ_α is an isomorphism, so is σ.

In particular we obtain an isomorphism $\widetilde{\sigma}\colon \mathcal{T}_2' \longrightarrow \mathbf{R}r_{2*}\mathbf{L}r_1^*\mathcal{T}_1^\vee$ by summing over $\alpha \in B_1$.

To define the desired natural transformation $\eta\colon \mathcal{F} \longrightarrow \mathsf{FM}$, we must construct a morphism

$$\eta(\mathcal{M})\colon \mathcal{T}_2' \overset{\mathbf{L}}{\otimes}_{E_1'} \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}_1}}(\mathcal{T}_1^\vee, \mathcal{M}) \longrightarrow \mathbf{R}r_{2*}r_1^*\mathcal{M}$$

for every $\mathcal{M}$ in $\mathcal{D}^b(\operatorname{coh}\mathcal{Z}_1)$. The desired map is the composition of

$$\mathcal{T}_2' \overset{\mathbf{L}}{\otimes}_{E_1'} \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}_1}}(\mathcal{T}_1^\vee, \mathcal{M}) \xrightarrow{\widetilde{\sigma}\otimes_{E_1'}\mathbf{R}r_{2*}\mathbf{L}r_1^*} \mathbf{R}r_{2*}\mathbf{L}r_1^*\mathcal{T}_1^\vee \overset{\mathbf{L}}{\otimes}_{E_1'} \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}_2}}(\mathbf{R}r_{2*}\mathbf{L}r_1^*\mathcal{T}_1^\vee, \mathbf{R}r_{2*}\mathbf{L}r_1^*\mathcal{M})$$

and the evaluation map from the derived tensor product to $\mathbf{R}r_{2*}\mathbf{L}r_1^*\mathcal{M}$. To show that η is an isomorphism, it suffices, since $\mathcal{T}_1^\vee$ generates, to prove that $\eta(\mathcal{T}_1^\vee)$ is an isomorphism. In this case, we have

$$\mathcal{T}_2' \overset{\mathbf{L}}{\otimes}_{E_1'} \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}_1}}(\mathcal{T}_1^\vee, \mathcal{T}_1^\vee) \cong \mathcal{T}_2' \overset{\mathbf{L}}{\otimes}_{E_1'} E_1' \cong \mathcal{T}_2' \cong \mathbf{R}r_{2*}r_1^*\mathcal{T}_1^\vee,$$

an isomorphism by construction. $\square$

Remark 5.5. Though we did not use it, in fact we have $E_1' \cong E_1$. Indeed, for $\alpha = (\alpha_1, \dots, \alpha_l) \in B_i$, define

$$\alpha^! = (n_i - l - \alpha_l, \dots, n_i - l - \alpha_1).$$

Then

$$\bigwedge^{\alpha'} \mathcal{Q}_i^\vee \cong \left(\bigwedge^l \mathcal{Q}_i\right)^{-(n_i-l)} \otimes_{\mathcal{O}_{\mathbb{G}_i}} \bigwedge^{(\alpha^!)'} \mathcal{Q}_i.$$

Thus

$$(\mathcal{T}_{\alpha,i})^\vee \cong p_i'^* \left(\bigwedge^l \mathcal{Q}_i\right)^{-(n_i-l)} \otimes_{\mathcal{O}_{\mathcal{Z}_i}} \mathcal{T}_{\alpha^!,i}$$

and hence

$$\mathcal{T}_i^\vee \cong p_i'^* \left(\bigwedge^l \mathcal{Q}\right)^{-(n_i-l)} \otimes_{\mathcal{O}_{\mathcal{Z}_i}} \mathcal{T}_i.$$

It follows that $\mathrm{End}_{\mathcal{O}_{\mathcal{Z}_i}}(\mathcal{T}_i^\vee) \cong \mathrm{End}_{\mathcal{O}_{\mathcal{Z}_i}}(\mathcal{T}_i)$.

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