Non-commutative desingularizations of determinantal varieties

joint work with R.-O. Buchweitz and M. Van den Bergh

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Motivation: Conjecture of Bondal-Orlov

Observation (Bondal-Orlov, ICM 2002)

If X is a smooth complex variety, and $\pi\colon \tilde{X} \longrightarrow X$ is a blowup of a smooth subvariety Y, then there is a natural fully faithful triangulated functor

$$\mathbf{L}\pi^*\colon \mathsf{D}^b(\operatorname{coh} X) \hookrightarrow \mathsf{D}^b(\operatorname{coh} \tilde{X})\,.$$

Proposal (BO)

Perhaps all the birational morphisms appearing in the minimal model program should also induce fully faithful inclusions on the derived level.

Motivation: Conjecture of Bondal-Orlov

Conjecture (BO)

If $\varphi\colon Y \dashrightarrow Y'$ *is a flop between smooth varieties, so fits into a diagram*

$$\begin{array}{ccc} Y & \overset{\varphi}{\dashrightarrow} & Y' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X & \end{array}$$

where

- f *and* f' *are resolutions of singularities with* $\operatorname{codim}(\textit{exc. locus}) \geq 2$; *and*
- *a technical condition on ampleness of divisors,*

then there is an equivalence

$$\mathsf{D}^b(\operatorname{coh} Y') \xrightarrow{\simeq} \mathsf{D}^b(\operatorname{coh} Y)\,.$$

Motivation: Conjecture of Bondal-Orlov

Example (2×2 Determinant)

Let $S = \mathbb{C}[x, y, u, v]$, and put $R = S/(xy - uv)$, a normal Gorenstein domain of dimension 3.

The hypersurface $\operatorname{Spec} R$ has two resolutions of singularities, obtained by blowing up either of the height-one primes

$$I = (x, u) \qquad \text{or} \qquad I' = (x, v)\,.$$

These blowups are related by a flop:

$$\begin{array}{ccc} \mathrm{Bl}_R(I) & \overset{\varphi = f' \circ f^{-1}}{\dashrightarrow} & \mathrm{Bl}_R(I') \\ & {\scriptstyle f} \searrow \qquad \swarrow {\scriptstyle f'} & \\ & \operatorname{Spec} R & \end{array}$$

Motivation: Conjecture of Bondal-Orlov

Bridgeland '03

Verified the conjecture of Bondal-Orlov in dimension 3.

Proposal (Van den Bergh '04)

Construct a non-commutative ring Λ such that there are equivalences

$$\mathsf{D}^b(\text{mod-}\Lambda)$$

$$\mathsf{D}^b(\mathrm{coh}\ Y) \xleftarrow{\ \simeq\ } \mathsf{D}^b(\text{mod-}\Lambda) \xrightarrow{\ \simeq\ } \mathsf{D}^b(\mathrm{coh}\ Y')$$

Example (2×2 Determinant, Redux)

$S = \mathbb{C}[x, y, u, v]$; $\quad R = S/(xy - uv)$; $\quad I = (x, u)$; $\quad I' = (x, v)$

Set

$$\begin{aligned} \Lambda &= \operatorname{End}_R(R \oplus I) \\ &\cong \begin{pmatrix} R & I \\ I' & R \end{pmatrix} \end{aligned}$$

Then

$$\mathsf{D}^b(\operatorname{coh} Y) \simeq \mathsf{D}^b(\text{mod-}\Lambda) \simeq \mathsf{D}^b(\operatorname{coh} Y').$$

Furthermore

- Λ has global dimension 3, and
- $\Lambda \cong R^2 \oplus I \oplus I'$ is maximal Cohen-Macaulay as an R-module.

Definition of NCCR

Definition (Van den Bergh '04)

Let R be a normal Gorenstein domain. A non-commutative crepant resolution of R (or of $\operatorname{Spec} R$) is an R-algebra Λ such that

- $\Lambda = \operatorname{End}_R(M)$ for a reflexive R-module M;
- Λ has finite global dimension; and
- Λ is a maximal Cohen–Macaulay R-module.

Equivalently, Λ is a symmetric, birational, non-singular R-order.

Van den Bergh recovered Bridgeland's result in some cases using NCCRs.

Determinantal varieties

Theorem (BLV)

Let $X = (x_{ij})$ *be an* $m \times n$ *generic matrix,* $t < m \leq n$*, and set*

$$R = k[X]/I_{t+1}(X)\,,$$

the quotient by the ideal of $(t+1) \times (t+1)$ *minors.*

Set

$$M \quad = \bigoplus_{\substack{\alpha \text{ partition} \\ m-t \text{ rows, } t \text{ cols}}} \operatorname{image}\left(\bigwedge^{\alpha_1} X^T \otimes \cdots \otimes \bigwedge^{\alpha_{m-t}} X^T\right)\,.$$

Then $\Lambda = \operatorname{End}_R(M)$ *is MCM and has finite global dimension.*
If $m = n$*, then* R *is Gorenstein and* Λ *is an NCCR.*

Construction

Lemma (Kapranov, BLV)

Let $\mathbb{G} = \mathrm{Grass}(t, k^m)$ *be the Grassmannian of* t*-planes in* k^m*. Let* $\mathcal{R}$ *be the tautological bundle on* $\mathbb{G}$*, and set*

$$\mathcal{T} \quad = \bigoplus_{\substack{\alpha \text{ partition} \\ m-t \text{ rows, } t \text{ cols}}} \bigwedge^{\alpha_1}\mathcal{R}^* \otimes \cdots \otimes \bigwedge^{\alpha_{m-t}}\mathcal{R}^* \,.$$

Then $\mathcal{T}$ *is a tilting bundle on* $\mathbb{G}$*.*

Proposition

Let $\quad \mathcal{Z} = \{(L, A) \in \mathbb{G} \times \mathrm{Mat}_{m\times n}(k) \mid \text{image } A \subseteq L\}$

be the Springer desingularization of $\mathrm{Spec}\, R$*. Then*

- $\pi_1^*\mathcal{T}$ *is a tilting bundle on* $\mathcal{Z}$*, and*
- $\pi_{2*}\pi_1^*\mathcal{T} = M$*.*

Quiver for the NCCR

$$M \quad = \bigoplus_{\substack{\alpha \text{ partition} \\ m-t \text{ rows, } t \text{ cols}}} \operatorname{image}\left(\bigwedge^{\alpha_1} X^T \otimes \cdots \otimes \bigwedge^{\alpha_{m-t}} X^T\right) .$$

So $\Lambda = \operatorname{End}_R(M)$ has one idempotent for each partition α fitting in a $(m-t) \times t$ box.

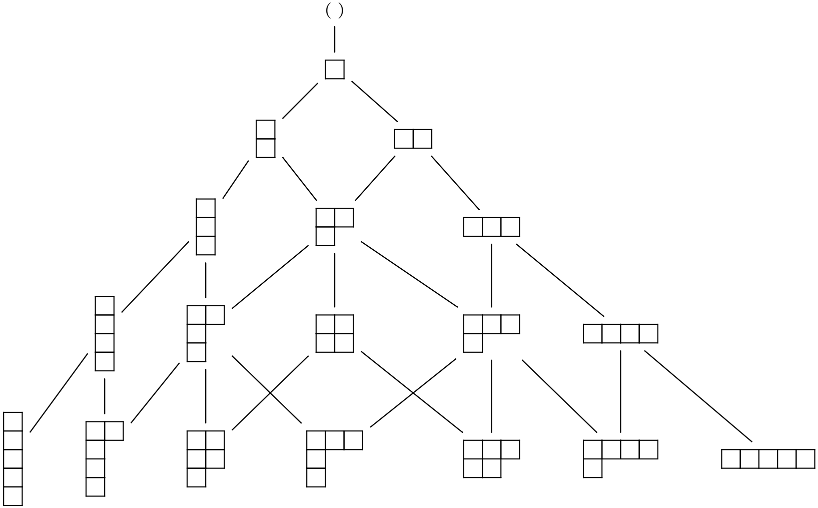
Lemma

The quiver defining Λ has arrows

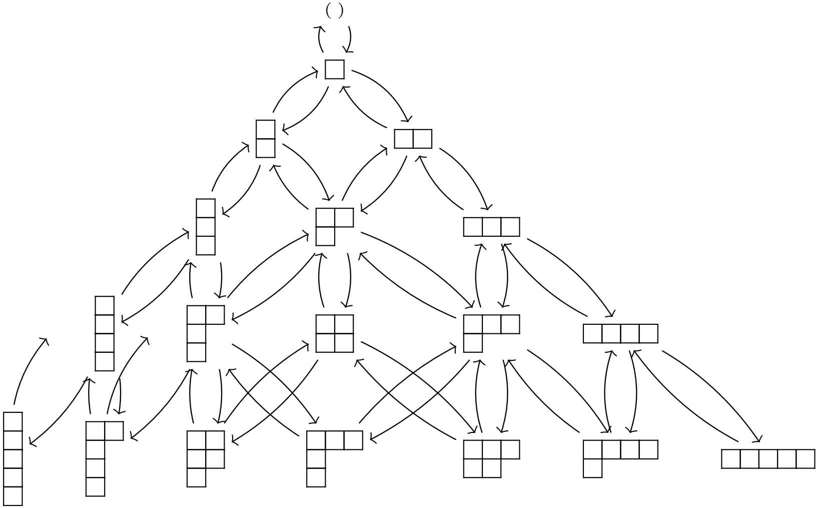
$$\alpha \underset{n}{\overset{m}{\rightleftarrows}} \beta$$

if β is obtained by adding a box to α.

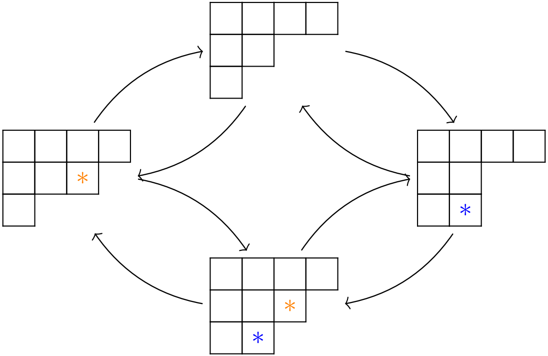
The Young Lattice



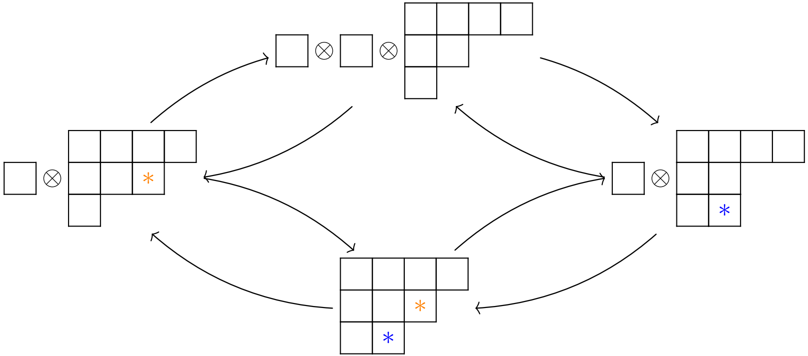
The Quiverized Young Lattice



Relations



Relations



Proposition

If k has characteristic zero and $t \geq m-2$, then the relations in the quiver are induced by forcing the Pieri maps to commute.