LOCAL RINGS OF COUNTABLE COHEN-MACAULAY TYPE

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ABSTRACT. We prove (the excellent case of) Schreyer's conjecture that a local ring with countable CM type has at most a one-dimensional singular locus. Furthermore we prove that the localization of a Cohen-Macaulay local ring of countable CM type is again of countable CM type.

Let $(R, \mathfrak{m})$ be a (commutative Noetherian) local ring of dimension d. Recall that a nonzero R-module M is called *maximal Cohen–Macaulay* (MCM) provided it is finitely generated and there exists an M-regular sequence $\{x_1, \dots, x_d\}$ in the maximal ideal $\mathfrak{m}$. We say that R itself is Cohen–Macaulay (CM) if it is MCM as a module over itself.

The CM local rings of finite CM-representation type (meaning that they have only finitely many nonisomorphic indecomposable MCM modules) have been carefully studied over the last twenty years. The complete equicharacteristic hypersurfaces of finite CM type have been completely classified ([6], [4], [9]), as have the complete equicharacteristic 2-dimensional normal domains ([2]). More generally, it is known that a CM local ring of finite CM type has at most an isolated singularity (proved by Auslander [1] in the complete case, Leuschke-Wiegand [10] in the excellent case, and Huneke-Leuschke [8] in general). Yoshino's monograph [15] is a comprehensive source for information about rings of finite CM type.

The related property of *countable* CM type has received much less attention. Buchweitz, Greuel, and Schreyer [4] classified the complete hypersurface singularities of countable CM type, but very little more has been learned since then.

The open questions and conjectures in Schreyer's 1987 survey article [12] have inspired work on both finite and countable CM-representation type. For example, Conjecture 7.3(a) states that a CM local ring R has finite CM type if and only if the $\mathfrak{m}$-adic completion has finite CM type; this was recently proved in case R is excellent in [10]. This paper is concerned with another of Schreyer's conjectures:

Conjecture 0.1 ([12])**.** *An analytic local ring over the complex numbers of countable CM type has at most a one-dimensional singular locus, that is, $R_{\mathfrak{p}}$ is regular for all primes $\mathfrak{p}$ with $\dim R/\mathfrak{p} > 1$.*

We verify Conjecture 0.1 more generally for all excellent CM local rings satisfying countable prime avoidance (Lemma 1.2). Some assumption of uncountability is necessary to avoid the degenerate case of a countable ring, which *a fortiori* has only countably many isomorphism classes of modules.

1. Schreyer's Conjecture

Definition 1.1. *A Cohen-Macaulay local ring $(R, \mathfrak{m})$ is said to have* finite (*resp.,* countable) Cohen–Macaulay type *if it has only finitely (resp., countably) many isomorphism classes of maximal Cohen–Macaulay modules.*

For the proof of Schreyer's conjecture, we need the following well-known lemma.

Lemma 1.2 (countable prime avoidance [5, Lemma 3]; see also [13])**.** *Let A be a Noetherian local ring which either is complete or has uncountable residue field. Let $\{\mathfrak{p}_i\}$, $i = 1, 2, \ldots$, be a countable family of prime ideals of A, $\mathfrak{a}$ an ideal of A, $x \in A$. Then $x + \mathfrak{a} \subseteq \mathfrak{p}_j$ for some j whenever $x + \mathfrak{a} \subseteq \bigcup_{i=1}^{\infty} \mathfrak{p}_i$.*

Theorem 1.3. *Let $(R, \mathfrak{m})$ be an excellent Cohen-Macaulay local ring of dimension d, and assume either that R is complete or that the residue field $R/\mathfrak{m}$ is uncountable. If R has countable CM type, then the singular locus of R has dimension at most one.*

Proof. Assume that the singular locus of R has dimension greater than one. Since R is excellent, this means that the singular locus is defined by an ideal J of height strictly less than $d - 1$. Let $\{M_i\}_{i=1}^{\infty}$ be a complete list of representatives for the isomorphism classes of indecomposable MCM R-modules. Consider the set

$$\Lambda = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} = \operatorname{Ann}(\operatorname{Ext}_R^1(M_i, M_j)), \text{for some } i, j, \text{ and } \dim(R/\mathfrak{p}) = 1\}.$$

Note that Λ is at most countable, and that J is contained in each $\mathfrak{p} \in \Lambda$. By countable prime avoidance (applied to R/J), the maximal ideal $\mathfrak{m}$ is not contained in the union of all $\mathfrak{p}$ in Λ, so there is an element $f \in \mathfrak{m} \setminus \bigcup_{\mathfrak{p} \in \Lambda} \mathfrak{p}$. Choose a prime $\mathfrak{q}$ containing f and J such that $\dim R/\mathfrak{q} = 1$; then of course $\mathfrak{q} \notin \Lambda$.

Let X (resp. Y) be a $(d-1)^{\text{th}}$ (resp. d^{th}) syzygy of $R/\mathfrak{q}$. Then X and Y are both MCM R-modules and we have a nonsplit short exact sequence

(*) $$0 \longrightarrow Y \longrightarrow F \longrightarrow X \longrightarrow 0,$$

where F is a free R-module. We claim that $\operatorname{Ann}(\operatorname{Ext}^1_R(X,Y)) = \mathfrak{q}$. It is clear that $\mathfrak{q}$ kills $\operatorname{Ext}^1_R(X,Y) \cong \operatorname{Ext}^{d+1}_r(R/\mathfrak{q},Y)$. To see the opposite containment, note that since $\mathfrak{q}$ contains J, $R_\mathfrak{q}$ is not regular. The resolution of the residue field of $R/\mathfrak{q}$ is thus infinite, and neither $X_\mathfrak{q}$ nor $Y_\mathfrak{q}$ is free, so (*) is nonsplit when localized at $\mathfrak{q}$.

We can write both X and Y as direct sums of copies of the indecomposables M_i, and further write

$$\operatorname{Ext}^1_R(X,Y) \cong \bigoplus_{i,j} \operatorname{Ext}^1_R(M_i, M_j)^{a_{ij}}$$

with all but finitely many of the a_{ij} equal to zero. Then $\mathfrak{q}$ is the intersection of the annihilators of the nonzero Ext modules appearing in the above decomposition. Since $\mathfrak{q}$ is prime, it must equal one of these annihilators, and then $\mathfrak{q} \in \Lambda$, a contradiction. □

2. Localization of rings with countable CM type

Let $(R,\mathfrak{m})$ be an excellent local ring of countable CM type, and assume R either has an uncountable residue field or is complete. By Theorem 1.3, the dimension of the singular locus of R is at most one. Thus there are at most finitely many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that $R_{\mathfrak{p}_i}$ is not regular and $\mathfrak{p}_i \neq \mathfrak{m}$. All such primes have dimension one, i.e., $\dim(R/\mathfrak{p}_i) = 1$ for $i = 1, \ldots, n$. To understand the structure of these rings, one wishes to know what type of singularity R has at these primes. A quick inspection of the list of examples given in [12] shows that $R_{\mathfrak{p}_i}$ has finite CM type! Our next main result shows that countable CM type localizes; hence in general each $R_{\mathfrak{p}_i}$ has countable CM type. Note also that in the case R is complete, while $R_{\mathfrak{p}_i}$ is no longer complete in general, it will have uncountable residue field.

Theorem 2.1. *Let $(R,\mathfrak{m})$ be a CM local ring with a canonical module. If R has countable CM type, then $R_\mathfrak{p}$ has countable CM type for each prime $\mathfrak{p}$ of R.*

Proof. Let $\mathfrak{p} \in \mathrm{Spec}(R)$ and assume that $\{M_\alpha\}$ is an uncountable family of R-modules such that $\{(M_\alpha)_\mathfrak{p}\}$ are nonisomorphic indecomposable MCM $R_\mathfrak{p}$ modules. For each α there is a *Cohen–Macaulay approximation* of M_α, that is, a short exact sequence

$$(\chi_\alpha) \qquad\qquad 0 \longrightarrow Y_\alpha \longrightarrow X_\alpha \longrightarrow M_\alpha \longrightarrow 0$$

wherein X_α is a MCM R-module and Y_α has finite injective dimension [3].

Since there are uncountably many modules X_α, there must be uncountably many X_α of some fixed multiplicity. Fixing that multiplicity, and using that there are only countably many isomorphism classes of MCMs, we then find that there are uncountably many short exact sequences

$$(\chi_\beta) \qquad\qquad 0 \longrightarrow Y_\beta \longrightarrow X \longrightarrow M_\beta \longrightarrow 0$$

where X is a fixed MCM R-module, Y_β has finite injective dimension, and the M_β are among our original list of M_α.

Since each $(M_\beta)_\mathfrak{p}$ is a MCM $R_\mathfrak{p}$-module and $(Y_\beta)_\mathfrak{p}$ has finite injective dimension over $R_\mathfrak{p}$, $\mathrm{Ext}^1_R(M_\beta, Y_\beta)_\mathfrak{p} \cong \mathrm{Ext}^1_R((M_\beta)_\mathfrak{p}, (Y_\beta)_\mathfrak{p}) = 0$. This follows from [11, Proposition 4.9]: if Y is a finitely generated module having finite injective dimension, then for all finitely generated R-modules M,

$$\mathrm{depth}(M) + \sup\{i\,|\,\mathrm{Ext}^i_R(M, Y) \neq 0\} = \mathrm{depth}(R).$$

In particular, each extension (χ_β) splits when localized at $\mathfrak{p}$. This implies that $(M_\beta)_\mathfrak{p}$ is a direct summand of $(X)_\mathfrak{p}$ for each β. But over a local ring, a finitely generated R-module Q can have at most finitely many non-isomorphic summands. [1] Since there are uncountably many $(M_\beta)_\mathfrak{p}$ which must be summands of $X_\mathfrak{p}$, this contradiction proves the theorem. $\qquad\square$

The results above, together with known examples, suggest a plausible question:

Let R be a complete local Cohen-Macaulay ring of countable CM type, and assume that R has an isolated singularity. Is R then necessarily of finite CM type?

We end the paper with an observation that having countable CM type descends from faithfully flat overrings. The method follows that of [14]. Countable CM type cannot in general ascend to the completion, since countable rings are of countable CM type, but their completions are not of

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[1]This can be seen by passing to the completion, where the Krull-Schmidt theorem holds. After completion, any direct summand of $\widehat{Q}$ must be isomorphic to a direct sum of a fixed finite subset of the indecomposable summands of the completion of Q. Hence there are only finitely many such isomorphism classes after completion, and since the completions of two modules are isomorphic if and only if the two modules are isomorphic, we are done (see [7, Proposition (2.5.8) Chap. IV, Section 2]).

countable CM type if the dimension of the singular locus is at least two. However, when the residue field is uncountable, we do not know if countable CM type ascends to the completion. The proof of the analogous assertion for finite CM type uses that the ring is necessarily Gorenstein on the punctured spectrum, which we do not know is true for countable CM type.

Proposition 2.2. *Let $(R, \mathfrak{m})$ be a CM local ring and $(S, \mathfrak{n})$ a faithfully flat R-algebra such that the closed fibre $S/\mathfrak{m}S$ is CM. If S has countable CM type, then so does R.*

Proof. Since the closed fibre is CM, the extension $S \otimes_R M$ of any MCM R-module is an MCM S-module. Let $Z_1, Z_2, \ldots$ be a complete list of all indecomposable MCM S-modules which are direct summands of $S \otimes_R M$ for some MCM R-module M (ignore any S-modules that do not appear in a direct-sum decomposition of some extended module). For each $i = 1, 2, \ldots$, choose an indecomposable MCM R-module X_i so that $Z_i \oplus W_i \cong S \otimes_R X_i$ for some S-module W_i.

For an indecomposable MCM R-module N, write $S \otimes_R N \cong \bigoplus_i Z_i^{a_i}$, where all but finitely many of the a_i are zero. We assume that $a_i = 0$ for $i > n$ and write the sum as a finite one. Then

$$(S \otimes_R N) \oplus W_1^{a_1} \oplus \cdots \oplus W_n^{a_n} \cong S \otimes_R (X_1^{a_1} \oplus \cdots \oplus X_n^{a_n}),$$

so $S \otimes_R N$ is a direct summand of $S \otimes_R (X_1^b \oplus \cdots \oplus X_n^b)$, where $b = \max\{a_i\}$. In other words, $S \otimes_R N$ is in the "plus category" of $S \otimes_R (X_1 \oplus \cdots \oplus X_n)$ (see [14]). By [14, Lemma 1.2], N is in the plus category of $X_1 \oplus \cdots \oplus X_n$, and by [14, Theorem 1.1], there are only finitely many possible such N. Since the set of all finite subsets of $\{X_1, X_2, \ldots\}$ is a countable set, this shows that R has only countably many indecomposable MCM modules up to isomorphism. $\square$

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