Semidualizing Modules and Gorenstein Presentations

joint work with D.A. Jorgensen and S. Sather-Wagstaff

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Urbana — 27 March 2009

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Definition of dualizing module

Standing assumptions throughout: $(R, \mathfrak{m}, k)$ is a Cohen–Macaulay local ring, and all modules are finitely generated.

Definition (Grothendieck '57)

Say that an R-module ω is dualizing if

- the natural map
  $$R \longrightarrow \mathrm{Hom}_R(\omega, \omega)\,,$$
  sending r to "multiplication by r," is an isomorphism;
- $\mathrm{Ext}^i_R(\omega, \omega) = 0$ for all $i > 0$; and
- ω has finite injective dimension.

Examples

Example

If R is Artinian, then $E_R(k)$, the injective hull of the residue field, is dualizing.

Example

The free module R is dualizing if and only if R is Gorenstein. (This can be taken as the definition of Gorensteinity.)

Example

There exist Cohen–Macaulay local rings R having no dualizing module [Ferrand-Raynaud '70]. (Precisely: there is a 1-dimensional local domain R such that the completion $\widehat{R}$ is not generically Gorenstein.)

Further properties

- If a dualizing module ω exists, then it is unique up to isomorphism.
- Whence the name “dualizing”:

Theorem

Let M be a maximal Cohen–Macaulay R-module. Then M is totally ω-reflexive, meaning that the assignment $M \rightsquigarrow M^\vee = \operatorname{Hom}_R(M, \omega)$ is a duality on MCM R-modules. Precisely,

- $\operatorname{Ext}^i_R(M, \omega) = 0$ *for all* $i > 0$,
- $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M, \omega), \omega) = 0$ *for all* $i > 0$, *and*
- *the natural map*

$$M \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega), \omega)$$

  is an isomorphism.

Existence of dualizing modules

Theorem (Sharp '71)

Suppose that R is a homomorphic image of a Gorenstein local ring Q. Then R has a dualizing module.

Sketch of proof.

Cut down by regular sequences to assume that Q and R are Artinian. Then $\mathrm{Ext}^i_Q(R, Q) = 0$ for all $i > 0$, and $\mathrm{Hom}_Q(R, Q)$ is dualizing for R. □

Consequences

- If R is complete then R has a dualizing module by the Cohen Structure Theorem.
- If R is essentially of finite type over a ring with a dualizing module, then R has a dualizing module.

A converse to Sharp's theorem

Theorem (Foxby '72, Reiten '72)

If R has a dualizing module ω, then R is a homomorphic image of a Gorenstein local ring Q.

Sketch of proof.

Let $Q = R \ltimes \omega$ be Nagata's idealization of ω, so:

- as R-modules, $R \ltimes \omega = R \oplus \omega$, and
- multiplication is given by

$$(r, c)(r', c') = (rr', rc' + r'c)\,.$$

With this structure, $R \ltimes \omega$ is a commutative local ring with maximal ideal $\mathfrak{m} \oplus \omega$, and surjects onto R via $(r, c) \mapsto r$. The ideal $0 \oplus \omega$ is square-zero.

From the fact that ω is R-dualizing, one checks that $R \ltimes \omega$ is Gorenstein. □

Definition of semidualizing module

Definition (Foxby '72, Vasconcelos '74, Golod '84, Wakamatsu '88)

Say that an R-module C is semidualizing (a.k.a. (PG), spherical, $G \ldots$) if

- the natural map $R \longrightarrow \mathrm{Hom}_R(C, C)$, sending r to "multiplication by r," is an isomorphism; and
- $\mathrm{Ext}^i_R(C, C) = 0$ for all $i > 0$.

Remarks

- A semidualizing module C is dualizing if and only if it has finite injective dimension.
- The rank-one free module R is always semidualizing.
- If R is Gorenstein, then the only semidualizing module is R. The converse holds if R has a dualizing module.

Further properties

- A semidualizing module C gives a duality only on the totally C-reflexive modules M, that is, the ones such that
  - $\mathrm{Ext}^i_R(M, C) = 0$ for all $i > 0$,
  - $\mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C) = 0$ for all $i > 0$, and
  - $M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism.

- Total reflexivity gives an order-type relation on the set of semidualizing modules: $C \trianglerighteq D$ if C is totally D-reflexive. Under this relation R is the maximum element, and a dualizing module ω is the minimum. It is not known whether this relation is transitive.

- If C is semidualizing and R has a dualizing module ω, then the dual $C^\vee = \mathrm{Hom}_R(C, \omega)$ is again a semidualizing module. If $C \ncong \omega, R$, then $C^\vee$ is never isomorphic to C.

Example of non-trivial semidualizing modules

Example (Foxby '87)

Set

$$A = k[\![x,y]\!]/(x,y)^2 \qquad \text{and} \qquad R = A[\![u,v]\!]/(u,v)^2\,.$$

Then both A and R are Artinian non-Gorenstein local rings. Put

$$C = \operatorname{Hom}_A(R, A)\,.$$

Then C is a semidualizing R-module. Since C has socle dimension 2, it is neither free nor dualizing. The same is true of $C^\vee = \operatorname{Hom}_R(C, \omega_R)$.

Notice that R has a non-trivial decomposition as a tensor product

$$R = \frac{k[\![x,y,u,v]\!]}{(x,y)^2 + (u,v)^2} \cong \frac{k[\![x,y,u,v]\!]}{(x,y)^2} \otimes_{k[\![x,y,u,v]\!]} \frac{k[\![x,y,u,v]\!]}{(u,v)^2}\,.$$

Existence of semidualizing modules

Theorem (DJ-GL-SSW '08, essentially Foxby '87)

Suppose Q is a Gorenstein local ring and $I_1, I_2 \subseteq Q$ are ideals. Set

$$R = Q/I_1 + I_2 = Q/I_1 \otimes_Q Q/I_2\,,$$

and assume that

- *each Q/I_j is Cohen–Macaulay and not Gorenstein;*
- *R is totally reflexive over each Q/I_j; and*
- *$\omega_R \cong D_1 \otimes_Q D_2$, where D_j is dualizing for Q/I_j.*

Then R admits a non-trivial semidualizing module, namely

$$R \otimes_{Q/I_j} D_j$$

for each j.

Existence of Gorenstein presentations

Theorem (DJ-GL-SSW '08)

Suppose that R has a dualizing module ω and a non-trivial semidualizing module C. Then there exists a Gorenstein local ring Q with ideals I_1, I_2 such that

- $R = Q/(I_1 + I_2) = Q/I_1 \otimes_Q Q/I_2$;
- *each* Q/I_j *is Cohen–Macaulay and not Gorenstein;*
- $\operatorname{Tor}_i^Q(Q/I_1, Q/I_2) = 0$ *for all* $i > 0$;
- R *is totally reflexive over each* Q/I_j;
- $D_1 \otimes_Q D_2 \cong \omega$ *and* $\operatorname{Tor}_i^Q(D_1, D_2) = 0$ *for* $i > 0$.

Proof of main theorem

$$
\begin{array}{ccc}
R_1 = R \ltimes C & & R_2 = R \ltimes C^\vee \\
& \searrow \quad \swarrow & \\
& R &
\end{array}
$$

Let R_1 be the idealization of C, and R_2 the idealization of $C^\vee$.

Then R_1 and R_2 are CM but not Gorenstein (as C, $C^\vee$ are not dualizing).

The dualizing modules for R_1 and R_2 are

$$D_1 = \operatorname{Hom}_R(R_1, \omega) \qquad \text{and} \qquad D_2 = \operatorname{Hom}_R(R_2, \omega)\,.$$

Proof cont'd

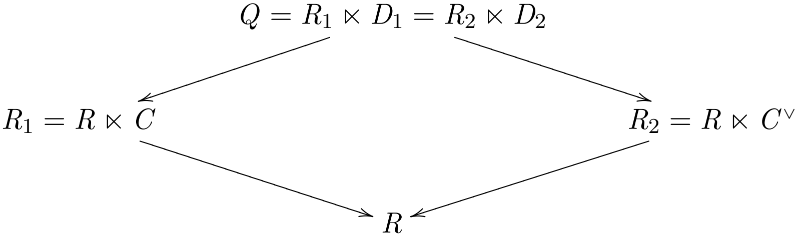
$$
\begin{array}{ccccc}
 & & Q = R_1 \ltimes D_1 = R_2 \ltimes D_2 & & \\
 & \swarrow & & \searrow & \\
R_1 = R \ltimes C & & & & R_2 = R \ltimes C^{\vee} \\
 & \searrow & & \swarrow & \\
 & & R & &
\end{array}
$$

The idealization of the R_1-module D_1 is a Gorenstein ring Q, which is isomorphic as an R-module to

$$Q \cong R \oplus C \oplus C^{\vee} \oplus \omega\,.$$

We show that Q is isomorphic (as a ring!) to $R_2 \ltimes D_2$.

Proof cont'd



To show $\operatorname{Tor}_i^Q(R_1, R_2) = 0$ for all $i > 0$, we work our way around the diamond, showing first that

$$R_1 \otimes_R R_2 \cong Q \qquad \text{and} \qquad \operatorname{Tor}_i^R(R_1, R_2) = 0\,,$$

then

$$Q \otimes_{R_1} R \cong R_2 \qquad \text{and} \qquad \operatorname{Tor}_i^{R_1}(Q, R) = 0\,,$$

and finally the desired vanishing.