Bounding the ranks of torsion-free modules on curves

Graham Leuschke, 11 November 03
at Queen's University

This is joint work with Roger Wiegand.

Notation:

- $(R, \mathfrak{m}, k)$ is a one-dimensional local ring
- always Cohen–Macaulay (so $\mathfrak{m}$ contains a nonzerodivisor)
- K is the total quotient ring (invert all the nonzerodivisors).
- $\widehat{R}$ is the completion, $\widetilde{R}$ is the integral closure in K
- An R-module M is *torsionfree* if it has depth one, or equivalently M embeds into a free module.
- Say M *has constant rank* if M is locally free of constant rank at the associated primes of R, or equivalently if $M \otimes K \cong K^r$. If this happens, then $\mathrm{e}(M) = r \cdot \mathrm{e}(R)$.

Definition 1. (1) Say R has *finite representation type* (FRT) if there are only finitely many indecomposable torsionfree R-modules, up to isomorphism.

(2) Say R has *bounded representation type* (BRT) if there is a bound on the multiplicities of indecomposble torsionfree R-modules.

Theorem 2. *(Drozd-Roĭter '67, Green-Reiner '78) TFAE:*

(1) R *has FRT*

(2) R *has BRT and* $\widehat{R}$ *is reduced*

(3) R *satisfies*

(DR1) $\mu_R(\widetilde{R}) \leq 3$

(DR2) $(\mathfrak{m}\widetilde{R} + R)/R$ *is a cyclic* R*-module.*

(4) *(Assume* R *contains a field.)* $\widehat{R}$ *birationally dominates one of the hypersurfaces* A_n, D_n, E_6, E_7, E_8.

Note: So, for example, you can write down all the complete domains of FRT that contain $\mathbb{C}$:

$$k[[t^2, t^n]], k[[t^3, t^4]], k[[t^3, t^5]], k[[t^3, t^4, t^5]], k[[t^3, t^5, t^7]]$$

A Word About The Proof:

• $(1) \implies (2)$ Obvious that FRT implies BRT. If $\widehat{R}$ is not reduced, then $\widetilde{R}$ is not finitely generated. So there are infinitely many rings S_i with $R \subset S_i \subset \widetilde{R}$. Since $\mathrm{End}_R(S_i) = S_i$, each is indecomposable and non-isomorphic, contradicting FRT.

• $(2) \implies (1)$ Follows from the "Brauer–Thrall Theorem" in the version Craig Huneke and I proved: BRT plus isolated singularity implies FRT (in any dimension).

• $(1) \implies (3)$ First, consider the Grassmannian of all 2-dimensional subspaces of $\widetilde{R}/\mathfrak{m}\widetilde{R}$. This has dimension $2(e-2)$. The group of units of $\widetilde{R}$ acts on X, and k^* acts trivially, so $\widetilde{R}^*/k^*$ acts on X. For any $x \in X$, the preimage in $\widetilde{R}$ is a torsionfree R-module, and one can show that distinct orbits of the group action give nonisormorphic modules. Since R has FRT, there are only finitely many orbits. So $\dim(X) \leq \dim(\widetilde{R}^*) - 1$, so $e \leq 3$. The other **DR** is similar – if $(\mathfrak{m}\widetilde{R} + R)/R$ contains linearly independent f and g, consider $S_\alpha := R[f + \alpha g]$.

• $(3) \implies (4)$ is grungy.

• $(4) \implies (1)$ needs AR quivers, which I won't talk about.

Note 2: So, for "analyically unramified" one-dimensional rings, FRT and BRT are equivalent. Today we'll talk about the case where $\widehat{R}$ is *not* reduced. Equivalently, $\widetilde{R}$ is not a finitely generated R-module.

Theorem 3. *Let k be an infinite field. Then the following is a complete list, up to isomorphism, of the complete one-dimensional CM local rings R with bounded but infinite representation type:*

(1) $A := k[[x,y]]/(y^2)$
(2) $T := k[[x,y]]/(xy^2)$
(3) $E := \operatorname{End}_T(\mathfrak{m}_T) \cong k[[x,y,z]]/(xy, yz, z^2)$

I'll prove this; it will take a while.

Proposition 4. *If R has multiplicity 2, then every indecomposable torsion-free R-module is an ideal.*

Proof (probably skip this): (Essentially Bass '63, improved by Rush '91 and L-Wiegand '03)
Bass proved that R is Gorenstein. In fact, every module-finite extension $R \subseteq S \subseteq K$ is Gorenstein. Let M be a nonfree indecomposable torsionfree R-module. First assume that M is faithful. Set

$$S := \{\alpha \in K \mid \alpha M \subseteq M\}.$$

Since M is faithful, $S \subseteq \operatorname{End}_R(M)$ is a module-finite ring extension of R. So S is isomorphic to an ideal of R.
If there is a surjection $M \to S$, then M has a direct summand isomorphic to S, so we're done.
If not, then $M^* := \operatorname{Hom}_S(M,S) = \operatorname{Hom}_S(M,\mathfrak{m}_S)$. That's a module over $E = \operatorname{End}_S(\mathfrak{m}_S)$, so M^{**} is as well. But $M^{**} \cong M$ since S is Gorenstein, so M is an E-module. Then $E = S$ by the definition of S. So S is a DVR. But then M is free, so is isomorphic to a direct sum of ideals.
If M is not faithful, pass to $R/\operatorname{Ann}_R(M)$ and repeat. $\square$

Note: This shows that $A = k[[x,y]]/(y^2)$ has BRT. The modules are the ideals $\{(x^i, y) \mid i = 0, 1, 2, \ldots, \infty\}$.

Next, multiplicity ≥ 4

Lemma 5. *If R has a module-finite extension S with $R \subseteq S \subseteq K$ and either*

(a) $\mu_R(S) \geq 4$ *or*
(b) $\mu_R(S) = 3$ *and* $\mu_R(\mathfrak{m}S/\mathfrak{m}) > 1$,

then R has indecomposable torsionfree modules of arbitrarily high constant rank.

Words about the proof: Consider $R' = R/\mathfrak{c}$ and $S' = S/\mathfrak{c}$, where $\mathfrak{c}$ is the conductor of S into R. Then R' and S' are an *Artinian pair*, and this result was proven for Artinian pairs by Dade in 1963. Roger showed how to lift back up to R and S in the late 80s.

Lemma 6. *A one-dimensional CM local ring R has an extension $R \subseteq S \subseteq K$ such that $\mu_R(S) = \mathrm{e}(R)$.*

Proof. Put $S_n = \mathrm{End}_R(\mathfrak{m}^n)$ and $S = \bigcup S_n$. To see that this works, we may assume k is infinite, and let x be a principal reduction of $\mathfrak{m}$. Then $x\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for $n \gg 0$, and we also have $\mu_R(\mathfrak{m}^n) = \mathrm{e}(R)$ for $n \gg 0$. Just check that for those n, $Sx^n = \mathfrak{m}^n$, so $\mu_R(S) = \mathrm{e}(R)$. $\square$

Corollary 7. *If $\mathrm{e}(R) \geq 4$, then R has indecomposable torsionfree modules of arbitrarily large constant rank.*

Next, multiplicity 3: $T = k[[x,y]]/(xy^2)$ and $E = \operatorname{End}_R(\mathfrak{m}_T)$.

Theorem 8 (BGS '87)**.** *Set $P = k[[x,y]]$, $T = P/(xy^2)$. Then every indecomposable torsionfree M has a presentation*

$$0 \longrightarrow P^n \xrightarrow{\varphi} P^n \longrightarrow M \longrightarrow 0$$

with φ one of the following:

$$(y), (x), (y^2), (xy), (xy^2)$$

$$\begin{pmatrix} y & x^k \\ 0 & -y \end{pmatrix}, \begin{pmatrix} xy & x^{k+1} \\ 0 & -xy \end{pmatrix}, \begin{pmatrix} xy & x^k \\ 0 & -y \end{pmatrix}, \begin{pmatrix} y & x^{k+1} \\ 0 & -xy \end{pmatrix}.$$

In particular, T has BRT.

Lemma 9. *Assume that R is Gorenstein (still dimension one), $E = \operatorname{End}_R(\mathfrak{m})$. Assume that E is local.*

(1) *Each nonfree indecomposable torsionfree R-module is also an E-module.*
(2) *Each indecomposable torsionfree E-module is a indecomposable tf R-module.*

Proof. (1) Since M has no free direct summands, $M^* = \operatorname{Hom}_R(M,R) = \operatorname{Hom}_R(M,\mathfrak{m})$. That's an E-module, so $M^{**} = M$ is as well.
(2) Any R-linear map $M \to M$ is also T-linear, so an idempotent over R is also idempotent over T. □

Now, we've show that A, T, E have BRT. Just need to show they're the only ones. Let R be a one-dimensional non-reduced complete local ring with BRT.

S'pose that $\mathrm{e}(R) = 3$. If $\mu(\mathfrak{m}) = 1$, done.

If $\mu(\mathfrak{m}) = 2$, then R is a hypersurface, $R = k[[x,y]]/(f)$. Since R is not reduced, we can write $f = g^3$ or $f = g^2 h$ with $g, h \in \mathfrak{m} \setminus \mathfrak{m}^2$. After a linear change of variables, we can write $f = y^3$ or $f = y^2(y+q)$, with $q \in xk[[x]]$.

case $f = y^3$**:** Put $S = R[\frac{y}{x}] = R + R\frac{y}{x} + R\frac{y^2}{x^2}$
Then $\mu_R(S) = 3$ and $\mu(\mathfrak{m}S/\mathfrak{m}) = 2$, so R doesn't have BRT.

case $f = y^2(y+q)$**,** $\mathrm{ord}_x(q) = 1$**:** This is T.

case $f = y^2(y+q)$**,** $\mathrm{ord}_x(q) \geq 3$**:** Then $R[\frac{y}{x}]$ is of the form $k[[x,z]]/(z^2(z+\frac{q}{x}))$. If R has BRT, then so does $R[\frac{y}{x}]$. So see the next case.

case $f = y^2(y+q)$**,** $q = x^2u$**:** Put $S = R[\frac{y}{x^2}, \frac{y^2+ux^2y}{x^5}]$.
Check that $\mu_R(S) = 3$, $\mu_R(\mathfrak{m}S/\mathfrak{m}) = 2$. Done with hypersurfaces.

If $\mu_{\mathfrak{m}}) = 3$, use a result of Sally: for a height-zero ideal J,

$$\mu(J) \leq \mathrm{e}(R) - \mathrm{e}(R/J).$$

Applied to $N = \mathrm{nil}(R)$, this gives $\mu_R(N) \leq 2$, and applied to N^2 it gives $N^2 = 0$. Fiddle around some more to see:

- if $\mu(N) = 2$, then $\mathfrak{m} = (x,y,z)$ with $x\mathfrak{m} = \mathfrak{m}^2$ and $N = (y,z)$
- if $\mu(N) = 1$, then $\mathfrak{m} = (x,y,z)$ with $x\mathfrak{m} = \mathfrak{m}^2$ and $N = Rz$ with $yz = z^2 = 0$.

If $\mu(N) = 2$, set $S = R[\frac{y}{x^2}, \frac{z}{x^2}]$.
Then $\mu(S) = 3$, $\mu(\mathfrak{m}S/\mathfrak{m}) = 2$, contradiction.

So $\mu(N) = 1$. Since $y^2 \in \mathfrak{m}^2 = x\mathfrak{m}$, we can write

$$y^2 = x^r(\beta y + \gamma z)$$

with $\beta, \gamma \in k[[x]]$, at least one a unit.

If $r \geq 2$, $S = R[\frac{y}{x^2}, \frac{z}{x^2}]$ does the trick again. So $r = 1$.

If β is a unit, set $B = k[[x]][y] \subseteq R$. Then $B \cong k[[x,y]]/(xy^2)$ and $R = \mathrm{End}_B(\mathfrak{m}_B)$. Done.

If γ is a unit, set $B = k[[x]][y+z]$ to get the same conclusion. Done. $\square$

The non-complete case:

Theorem 10. *Let R be one-dimensional equicharacteristic CM local, infinite residue field. Then R has BRT iff $\widehat{R}$ has BRT.*

Note: Ascent of BRT is relatively easy. If $\widehat{R}$ does not have BRT, then by the main theorem it has indecomposable torsionfree modules of unbounded constant rank. Then use:

Lemma 11. *A torsionfree $\widehat{R}$-module which is locally free at the minimal primes is extended from R if and only if $\mathrm{rk}_{\widehat{R}_p}(M_p) = \mathrm{rk}_{\widehat{R}_q}(M_q)$ for all $p, q \in \mathrm{Min}(\widehat{R})$ with $p \cap R = q \cap R$. In particular, modules of constant rank are always extended.*

Note: Descent of BRT is much harder. Luckily, [BGS] *classified* the modules over T, and therefore over E. That, plus the Lemma and some combinatorics, finishes descent in dimension one.

Connections with CRT: Why did [BGS] care about the modules over T and E? They were proving:

Theorem 12. *The complete equichar hypersurfaces of countable representation type are*

- $A_\infty = k[[x, y, z_1, \ldots, z_n]]/(y^2 + z_1^2 + \cdots + z_n^2)$
- $D_\infty = k[[x, y, z_1, \ldots, z_n]]/(xy^2 + z_1^2 + \cdots + z_n^2)$

So we see that for one-dimensional hypersurfaces, BRT iff CRT. In fact,

Theorem 13 (L-Wiegand)**.** *The complete equichar hypersurfaces of BRT are*

- $A_\infty = k[[x, y, z_1, \ldots, z_n]]/(y^2 + z_1^2 + \cdots + z_n^2)$
- $D_\infty = k[[x, y, z_1, \ldots, z_n]]/(xy^2 + z_1^2 + \cdots + z_n^2)$

So for hypersurfaces, BRT iff CRT. Note that E also has CRT, but hardly any other rings of CRT are known. What does all this mean?