Branched Covers and Matrix Factorizations

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Matrix factorizations

Observe that $f = xy + z^2$ is irreducible over any field k. However,

$$\begin{bmatrix} xy+z^2 & \\ & xy+z^2 \end{bmatrix} = \begin{bmatrix} x & z \\ -z & y \end{bmatrix} \begin{bmatrix} y & -z \\ z & x \end{bmatrix}.$$

We call such a thing a matrix factorization of f.

Definition (Eisenbud '80)

Let S be a (regular local) ring and $f \in S$. A matrix factorization (MF) of f is a pair of $n \times n$ matrices (φ, ψ) such that

$$\varphi\psi = f \cdot I_n = \psi\varphi.$$

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MFs and MCM modules

Theorem (Eisenbud, '80)

The correspondence

$$(\varphi, \psi) \longleftrightarrow \operatorname{coker} \varphi$$

defines a bijection between reduced MFs of f (up to a natural equivalence relation) and maximal Cohen-Macaulay (MCM) R-modules with no free direct summands (up to isomorphism).

Example

- $(f, 1) \longmapsto \operatorname{coker} f = R$
- $(1, f) \longmapsto \operatorname{coker} 1 = 0$

Call a matrix factorization reduced if it has no unit entries.

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The double branched cover

Given $f \in S$ and $R = S/(f)$, we define

$$R^{\#} = S[[z]]/(f + z^2)\,,$$

the double branched cover of R.

There is a functor sending MFs of f to MFs of $f + z^2$, defined by

$$(\varphi, \psi) \longmapsto \left(\begin{bmatrix} \psi & -z \\ z & \varphi \end{bmatrix}, \begin{bmatrix} \varphi & z \\ -z & \psi \end{bmatrix} \right)$$

One can show that on MCM modules this corresponds to

$$\operatorname{coker}\varphi \longmapsto \operatorname{syz}^{R^{\#}}(\operatorname{coker}\varphi)\,.$$

We also have a reverse functor, killing z.

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Knörrer's results

$R = S/(f)$, $R^{\#} = S[[z]]/(f + z^2)$

Theorem (Eisenbud '80, Knörrer '87)

Assume $\operatorname{char} k \neq 2$. Then the following are equivalent:

1. *There are only finitely many indecomposable MCM R-modules up to $\cong$ (finite CM type).*
2. *There are only finitely many indecomposable MFs of f (finite MF type).*
3. *There are only finitely many indecomposable MCM $R^{\#}$-modules up to $\cong$.*
4. *There are only finitely many indecomposable MFs of $f + z^2$.*

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Classification of finite CM type

Theorem (Buchweitz-Greuel-Schreyer, Knörrer '87)

Let k be an algebraically closed field of characteristic zero and let $R = k[[x_1, \ldots, x_r]]/(f)$, where $f \in (x_1, \ldots, x_r)^2$ is nonzero. Then R has finite CM type if and only if R is a simple hypersurface singularity, that is, after a change of variables f is among the following:

(A_n) $x^2 + y^{n+1} + x_3^2 + \cdots + x_r^2$, *for* $n \geq 1$

(D_n) $x^2 y + y^{n-1} + x_3^2 + \cdots + x_r^2$, *for* $n \geq 4$

(E_6) $x^3 + y^4 + x_3^2 + \cdots + x_r^2$

(E_7) $x^3 + xy^3 + x_3^2 + \cdots + x_r^2$

(E_8) $x^3 + y^5 + x_3^2 + \cdots + x_r^2$

d-fold MFs

From now on, fix an integer $d \geq 2$ and assume $\operatorname{char} k \nmid d$.

Definition

A *d-fold matrix factorization* of f is a d-tuple of square matrices $(\varphi_1, \ldots, \varphi_d)$ with

$$\varphi_1 \cdots \varphi_d = f \cdot I_n \, .$$

Get a (exact, Frobenius) category $\mathrm{MF}^d(f)$.

First (?) considered by Backelin-Herzog-Sanders '06 for homogeneous f of degree d.

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The d-fold Branched Cover

We still have a correspondence with MCM modules via

$$(\varphi_1, \ldots, \varphi_d) \longmapsto \operatorname{coker} \varphi_1 \,.$$

Question

The modules corresponding to d-fold MFs form a subcategory of $\mathrm{MCM}(R)$. Which ones are they?

Answer

We relate them to the d-fold branched cover defined by $f + z^d$.

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Extending Knörrer

Theorem (L-Tribone)

The following are equivalent:

(1) *There are only finitely many indecomposable d-fold MFs of f.*

(2) *The d-fold branched cover $R^\# = S[[z]]/(f + z^d)$ has finite CM type.*

Proof Idea.

Given $N \in \mathrm{MCM}(R^\#)$, let $\varphi\colon N \longrightarrow N$ be the S-linear map given by multiplication by z. Since $z^d = -f$, get a d-fold MF $N^\flat = (\varphi, \dots, \varphi) \in \mathrm{MF}^d(f)$ (up to sign).

Given a d-MF $(\varphi_1, \dots, \varphi_d)$ of size n, define a MCM $R^\#$-module structure on R^{nd} with z-action given by the φ_i.

These aren't inverses, but the composition is understandable. $\square$

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Finite d-MF Type

Theorem (L-Tribone)

Assume $d > 2$. Let k be an algebraically closed field of characteristic zero and let $f \in S = k[[y, x_2, \ldots, x_r]]$. Then f has finitely many indecomposable d-fold MFs if and only if, after a change of variables, f and d are among:

(A_1) $y^2 + x_2^2 + \cdots + x_r^2$, for any $d > 2$

(A_2) $y^3 + x_2^2 + \cdots + x_r^2$, for $d = 3, 4, 5$

(A_3) $y^4 + x_2^2 + \cdots + x_r^2$, for $d = 3$

(A_4) $y^5 + x_2^2 + \cdots + x_r^2$, for $d = 3$

For each of these, we can write down the indecomposable d-MFs explicitly using, for example, the classification of MCMs over $z^d + y^2 + x_2^2 + \cdots + x_r^2$.

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Reduced d-MFs

Fact

If (φ, ψ) is a 2-fold matrix factorization and any entry of either matrix is a unit, then either $(1, f)$ or $(f, 1)$ is a direct summand.

Equivalently, (φ, ψ) is reduced if and only if $\operatorname{coker} \varphi$ and $\operatorname{coker} \psi$ have no free direct summands.

This is not true for $d > 2$.

Example

$$X = \left(\begin{bmatrix} x & y \\ 0 & -x \end{bmatrix}, \begin{bmatrix} 0 & y \\ x^2 & -x \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ x & y \end{bmatrix} \right)$$

is an indecomposable 3-fold matrix factorization of x^2y.

So which MCM modules correspond to the reduced d-MFs?

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Reduced d-MFs, cont'd

Lemma

Let N be an MCM $R^{\#}$-module and assume that $f + z^d$ is irreducible (so $R^{\#}$ is a domain). Then

$$\mu_{R^{\#}}(N) \leq d \cdot \operatorname{rank}_{R^{\#}}(N)\,,$$

and equality holds if and only if the S-linear map $z\colon N \longrightarrow N$, considered as a matrix over S, is reduced.

Proof Idea.

The S-module N is free, and the map $z\colon N \longrightarrow N$ is a free presentation of the S-module N/zN. It's a minimal presentation if and only if the matrix is reduced. □

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Ulrich modules

Fact/Definition (Brennan-Herzog-Ulrich '87)

An MCM module M over a domain A always satisfies

$$\mu_A(M) \leq e(A) \cdot \operatorname{rank}_A(M)\,,$$

where $e(A)$ is the multiplicity. If equality holds we say M is an Ulrich module (or a maximally generated MCM module).

Corollary

Assume that $\operatorname{ord}(f) \leq d$ (so that the multiplicity of $R^{\#}$ is equal to d). Then an MCM $R^{\#}$-module N is Ulrich if and only if the corresponding d-MF $N^{\flat}$ is reduced.

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Ulrich modules, cont'd

Using the classification of hypersurfaces of finite d-MF type, together with a classification of Ulrich modules over $k[[x,z]]/(x^{n+1}+z^n)$ [Herzog-Ulrich-Backelin '91], we can show:

Example

For $n \geq 4$, the polynomial $x^{n+1} \in k[[x]]$ has infinitely many indecomposable matrix factorizations with n factors, but only finitely many that are reduced, namely $(x^2, x, \ldots, x)$ and its shifts.

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