MATRIX FACTORIZATIONS AND KNÖRRER PERIODICITY

VIRTUAL COMMUTATIVE ALGEBRA SEMINAR, IIT BOMBAY

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BACKGROUND

Knörrer: relate MFs of f and of $f+z^2$

New results at the end, due to Tribone

Observe: $xy+z^2$ is irreducible (char $\neq 2$) But

$$\begin{bmatrix} xy+z^2 & 0 \\ 0 & xy+z^2 \end{bmatrix} = \begin{bmatrix} x & z \\ -z & y \end{bmatrix} \begin{bmatrix} y & -z \\ z & x \end{bmatrix}$$

(Known to Dirac 1930s)

Def Let S be a regular local ring and $f \in S$. Then a matrix factorization (MF) of f is a pair of matrices (φ, ψ) s.t.

$$\boxed{\varphi\psi = \psi\varphi = f \cdot I_n}$$

MFs and modules: Since f is nzd, both φ and ψ are 1-1:

$$0 \to S^n \xrightarrow{\varphi} S^n \to \operatorname{cok}\varphi \to 0$$

$f(S^n) = \varphi\psi(S^n) \subseteq \operatorname{im}\varphi$, so f kills $\operatorname{cok}\varphi$. That is, $\operatorname{cok}\varphi$ is a module over the hypersurface ring $R = S/(f)$.

Recall ① A f.g. module M over a local ring A is called maximal Cohen-Macaulay (MCM) if

$$\operatorname{depth}_A(M) = \dim A \quad \text{(max poss.)}$$

(2) (Auslander–Buchsbaum) If $\mathrm{pd}_A M < \infty$
then
$$\operatorname{depth}_A M = \operatorname{depth} A - \underline{\underline{\mathrm{pd}_A M}}.$$

Putting all this together: $\operatorname{cok}\varphi$ is a MCM R-module

<u>Conversely</u>, any MCM R-module has proj dim 1 over S, so
$$0 \to S^n \xrightarrow{\varphi} S^n \to M \to 0$$
a free res'n. Mult. by f kills M, so is homotopic to 0 on this res'n, so we get $\psi : S^n \to S^n$ so that
$$\varphi\psi = f \cdot I_n$$

<u>Thm</u> [Eisenbud '80] The correspondence
$$(\varphi, \psi) \longleftrightarrow \operatorname{cok}\varphi$$
defines a bijection between <u>reduced</u> MFs of f (up to a

natural equiv. rel) and MCM
R-modules without free summands.

eg: $\cdot$ $(f, 1) \longleftrightarrow \operatorname{cok} f = R$

$\cdot$ $(1, f) \longleftrightarrow \operatorname{cok} 1 = 0$

Say (φ, ψ) is reduced if it has no summand of either of these.

Easy application: $R = k[x]/x^2$ - the indec. modules are R and k

MFs $(x^2, 1)$ and (x, x)

What about $R = k[x,y]/(x^2)$?

[BGS '87] : the only indec. MCM mods are R, $R/(x)$, and

$$\operatorname{cok} \begin{bmatrix} x & y^n \\ 0 & -x \end{bmatrix} \quad n \geq 1$$

New phenomenon: Given two MFs (φ, ψ) of f, (φ', ψ') of g, in disjoint sets of variables, get

$$\left(\begin{bmatrix} \psi & -\varphi' \\ \psi' & \varphi \end{bmatrix}, \begin{bmatrix} \varphi & \varphi' \\ -\psi' & \psi \end{bmatrix} \right)$$

a MF of $f+g$. (in combined sets of variables)

Yoshino calls this <u>tensor product</u>

$$(\varphi,\psi) \,\hat{\otimes}\, (\varphi',\psi')$$

<u>Important</u>: this has nothing to do with the usual $\otimes$ of modules

$$\operatorname{cok}\varphi \otimes \operatorname{cok}\varphi'$$

(Mathematical physicists love these.)

<u>Special case</u>: $g = z^2$. We get

$$R = S/(f) \quad \text{and} \quad R^\# = S[z]/(f+z^2)$$

(the <u>double branched cover</u>)

$$(\varphi,\psi) \longmapsto (\varphi,\psi)\,\hat{\otimes}\,(z,z)$$

$$\left(\begin{bmatrix} \psi & -z \end{bmatrix} \quad \begin{bmatrix} \varphi & z \end{bmatrix} \right)$$

$MF(f) \qquad = \left(\begin{bmatrix} & \\ zI & \varphi \end{bmatrix}, \begin{bmatrix} & \\ -z & \psi \end{bmatrix} \right)$

At the level of MCM modules,

$$\operatorname{cok}\varphi \longmapsto \Omega_{R^\#}(\operatorname{cok}\varphi)$$

$\uparrow$ mod-$R \subseteq$ mod $R^\#$ (syzygy)

We also have a reverse functor, killing z.

<u>Knörrer</u> a) $\overline{(\varphi,\psi)\hat{\otimes}(z,z)} = (\varphi,\psi)\oplus(\psi,\varphi)$

b) $\overline{(\Phi,\Psi)}\hat{\otimes}(z,z) = (\Phi,\Psi)\oplus(\Psi,\Phi)$

c) f has only finitely many indec MFs iff $f+z^2$ does too.

<u>Rmk</u>: (c) fails for $f+z^d$, $d\geq 3$

$\Longrightarrow$ Classification of hyp's of fCMt.

Key Ingredient: related $MF(f)$ with

the skew group algebra $R^\#[\sigma]$

Here $R = S/(f)$

$R^\# = S[z]/(f+z^2)$

σ autom of $R^\#$, fixes S and $\sigma(z) = -z$.

$R^\#[\sigma]$ has elements $a \cdot 1 + b\sigma$, with $\sigma^2 = 1$ and $\sigma \cdot a = \sigma(a)\cdot\sigma$

Knörrer: There is an equivalence

$$MF(f) \simeq MCM(R^\#[\sigma])$$
$$= \{R^\#[\sigma]\text{-mods free} / S\}$$

Moral: MFs of f "already know" about the double branched cover.

New results [T. Tribone]

$MF^d(f)$ = d-fold MFs

$(\varphi_1, \varphi_2, \ldots, \varphi_d)$

$\varphi_1\varphi_2 \cdots \varphi_d = f \cdot I_n$

First (?) considered by Bäckelin–Herzog–Sanders '90 (Childs '78)

$\longrightarrow$ If f is homog of deg d each φ_i is linear, so $\operatorname{cok}\varphi_i$ is an <u>Ulrich</u> module

<u>Obs</u> ① Given $(\varphi,\psi)\in MF^2(f)$, can <u>inflate</u> it

$$(\varphi,\psi,1,1,\_,1)\in MF^d(f)$$

These are boring.

eg $(x,y,z)\in MF^3(xyz)$

② Given $(\varphi_1,\varphi_2,\ldots,\varphi_d)\in MF^d(f)$ can <u>collapse</u>

$$(\varphi_1,\varphi_2\cdots\varphi_d)\in MF^2(f)$$

Collapsing corresponds to <u>extensions</u> of the MCM modules eg $(\varphi_1, \varphi_2, \varphi_3) \longrightarrow (\varphi_1, \varphi_2\varphi_3)$

$$\Rightarrow 0 \to \operatorname{cok}\varphi_2 \to \operatorname{cok}(\varphi_2\varphi_3) \to \operatorname{cok}\varphi_3 \to 0$$

$$\operatorname{cok}(\varphi_2\varphi_3) \cong \Omega(\operatorname{cok}\varphi_1)$$

Thm [Tribone]

$$MF^d(f) \simeq MCM(R^\#[\sigma])$$

where

$$R = S/(f)$$

$$R^\# = S[z]/(f+z^d)$$

$\sigma \in \operatorname{Aut}(R^\#)$, fixes S,

$$\sigma(z) = \omega z \quad (\omega^d = 1)$$

Oh no! Lost network!