Splitting properties of big Cohen-Macaulay modules

Mel Hochster, Graham Leuschke, Rebecca R. G.

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Artin algebras

Recall that an artinian ring Λ is said to have finite representation type if there are only finitely many indecomposable finitely generated left Λ-modules up to isomorphism.

Theorem (Ringel-Tachikawa, Auslander '74)

Let Λ be an artinian ring of finite representation type. Then every left Λ-module is a direct sum of finitely generated Λ-modules.

In fact, Auslander also proved the converse: if every module is a sum of finitely generated modules, then Λ has finite representation type.

Integral group representation theory

Definition

- Let T be a Dedekind domain. A T-order is a T-algebra Λ which is finitely generated projective as a T-module.
- A lattice over Λ is a finitely generated Λ-module which is projective over T.
- A generalized lattice is the same thing without assuming finite generation.

Example

$\widehat{\mathbb{Z}}_p G$, the p-adic group ring of a finite group G.

Integral group representation theory

Definition

Let T be a DVR and Λ a T-order. Say that Λ has finite lattice type if there are only finitely many indecomposable lattices up to isomorphism.

Theorem (Butler-Campbell-Kovács '04)

Let Λ be a T-order of finite lattice type. Then every generalized lattice is a direct sum of lattices.

Example

If all Sylow p-subgroups of G are cyclic of order p or p^2, then every generalized lattice over $\widehat{\mathbb{Z}}_p G$ is a direct sum of lattices.

MCM representation theory

Goal

Consider analogues of the results of Auslander, Ringel-Tachikawa, and Butler-Campbell-Kovács for big Cohen-Macaulay modules in dimensions ≥ 2.

From now on, let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay complete local ring.

Definition

- A finitely generated R-module M is maximal Cohen-Macaulay (MCM) if every system of parameters in $\mathfrak{m}$ is a regular sequence on M.
- A balanced big Cohen-Macaulay (big CM) module is the same thing without assuming finite generation.
- Say R has finite CM type if there are only finitely many indecomposable MCM modules up to isomorphism.

Previously

Theorem (Holm '14)

Every big CM module over a CM complete local ring $(R, \mathfrak{m}, k)$ is a direct limit of MCM modules.

Note the similarity with the theorem of Govorov-Lazard: a module M over a commutative ring A is flat iff it is a direct limit of finitely generated free modules.

Corollary

Over a complete RLR, big CM $\iff$ flat.

Extending Butler-Campbell-Kovács

Assume that R contains a complete regular local ring T over which it is finite. (Could say that R is a T-order.)

Definition

- A lattice over R is a finitely generated R-module which is projective over T.
- A generalized lattice is the same thing without assuming finite generation.

Easy Observations

- lattice $\iff$ MCM
- generalized lattice $\implies$ big CM, but not conversely.

Main result 1

Theorem

Let $T \subseteq R$ be a finite extension of two-dimensional complete local rings with T regular. Assume that R has finite lattice type. Then every generalized lattice over R is a direct sum of lattices.

Lemma

Let B be a big CM R-module and B' a first syzygy of B. Then B' is a lattice.

Corollary

The first syzygy of every big CM R-module is a direct sum of MCM modules.

Proof sketch

The proof follows the outlines of Butler-Campbell-Kovács.

- Let $G = \bigoplus M_i$ be the Auslander module, which generates $\mathrm{MCM}(R)$. Set $\Lambda = \mathrm{End}_R(G)$, the Auslander algebra. Then Λ has global dimension 2.
- The functor $\mathrm{Hom}_R(G, -)\colon$ Mod-$R \longrightarrow$ Mod-Λ defines an equivalence

$$\{\text{big CM } R\text{-modules}\} \xrightarrow{\sim} \{\text{flat } \Lambda\text{-modules}\}$$

- The restriction gives an equivalence

$$\{\text{generalized } R\text{-lattices}\} \xrightarrow{\sim} \{\text{projective } \Lambda\text{-modules}\}$$

- Use the equivalence to show that any generalized lattice is a direct summand of a direct sum of copies of G. $\square$

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McKay Correspondence

Another approach uses Auslander's MCM McKay Correspondence.

Theorem (Auslander '86)

Suppose R is a complete local $\mathbb{C}$-algebra of dimension 2, and R has finite CM type. Then $R \cong \mathbb{C}[\![u,v]\!]^G$ for some finite subgroup $G \subset \mathrm{GL}_2(\mathbb{C})$ acting linearly.

Theorem (Herzog '87)

In the above situation, the MCM R-modules are precisely the R-summands of $\mathbb{C}[\![u,v]\!]$.

Invariant theory

Fact (from classical invariant theory)

Let $S = \mathbb{C}[\![u, v]\!]$ and $R = S^G$, where G is a finite group.

Then the R-direct summands of S are (up to multiplicities) the isotypic components S_ρ of the action. Here ρ runs over all irreducible representations of G.

In particular every indecomposable MCM R-module M is isomorphic to some S_ρ, and

$$\operatorname{rank} S_\rho = \dim_k \rho \, .$$

Main result 2

Theorem

Let $R \subseteq S$ be a finite extension of two-dimensional complete local rings with S regular. Assume that R is a summand of S as R-modules, and that S is isomorphic to a direct sum of R-modules of rank 1. Then every big CM R-module is a direct sum of MCM modules.

Corollary

If $R = k[\![u, v]\!]^G$, where G is an abelian group of order invertible in k, then every big CM R-module is a direct sum of MCM modules.

In particular this includes the (A_n) singularities $k[\![x, y, z]\!]/(xy - z^{n+1})$.

Recent other work

Theorem (Bahlekeh-Fotouhi-Salarian '18)

A complete CM local ring R is of finite CM type if and only if every balanced big CM R-module M with $\mathfrak{m}$-primary cohomological annihilator is a direct sum of finitely generated modules.

I don't know whether this result subsumes ours, but I suspect so.