"Knörrer periodicity and a generalization"

joint w/ Alex Dugas.

<u>Setup</u> k field

$S = k[[x_0, \_, x_d]]$

$f \in S$ nonzero nonunit.

$R = S/(f)$ d-dim'l hypersurface ring

Recall: MCM $\leftrightarrow$ MF + all resn's eventually 2-periodic

<u>Def</u> $R^\# = S[y]/(f + y^n)$, the <u>n-fold branched cover of R</u> (of Spec S branched over Spec R).

<u>Goal</u>: relate the MCM modules / $R, R^\#$.

We have a natural map $R^\# \twoheadrightarrow R$ given by $y \mapsto 0$ (no map the other direction in general). So we get two functors

$$\mathrm{MCM}(R^\#) \to \mathrm{MCM}(R), \quad \mathrm{MCM}(R) \to \mathrm{MCM}(R^\#)$$
$$N \longmapsto N/yN \qquad\qquad M \longmapsto \Omega_{R^\#}(M).$$

These are especially well-behaved when $n=2$.

'87

<u>Theorem</u> [Knörrer]. Let $n=2$. Assume char $k \neq 2$. Then for any

- MCM R-mod M,
  $\Omega_{R^\#}(M)/y\Omega_{R^\#}(M) \cong M \oplus \Omega_R(M)$.
- MCM $R^\#$-mod N,
  $\Omega_{R^\#}(N/yN) \cong N \oplus \Omega_{R^\#}(N)$.

In particular, R has <u>finite representation type</u> (i.e. only fin many indecomposable MCM modules up to $\cong$) iff $R^\#$ does.

<u>Consequence</u> [Buchweitz-Greuel-Schreyer '87] If char $k=0$, $k=\bar{k}$ then $R=S/(f)$ has FRT iff

$$f \cong g(x_0,x_1) + x_2^2 + \cdots + x_d^2$$

where g is one of

A_n	$x^2 + y^{n+1}$
D_n	$x^2 + xy^{n-1}$
E_6	$x^3 + y^4$
E_7	$x^3 + xy^3$
E_8	$x^3 + y^5$.

<u>Pf sketch</u>: We know FRT $\Rightarrow$ simple.

<u>Lemma</u> simple + dim $\geq 2 \Longrightarrow e(R) \leq 2$

so we can use Weierstrass preparation to write $f(x_0,\ldots,x_d) = g(x_0,\ldots,x_{d-1}) + x_d^2$.

Then $S/(f) = \overset{k[x_0, \dots, x_{d-1}]}{\cancel{S}[x_d]} \Big/ g + x_d^2 = R^\#$ has FRT iff ~~$R = S/f$~~, $R = k[x_0, \dots, x_{d-1}]/g$ does. Repeat to get down to dim 1 where we know the classification. ✓ $\square$.

Now let $n \geq 2$.
We hope for something similar.

Theorem [Herzog-Popescu '97] Let $n \geq 2$. Assume char $k \nmid n$. For any MCM $R^\#$-module N,

$$\Omega_{R^\#}(N/y^{n-1}N) \cong N \oplus \Omega_{R^\#}(N)$$

(*Notice* $N/y^{n-1}N$ is *not* an R-module, but a module over the CI $S[y]/(f, y^{n-1})$.)

Key Lemma: Let Q be any ring, X a Q-module, and $z \in Q$ a nzd on X. Then $\exists$ SES

$$0 \to \Omega_Q(X) \longrightarrow \Omega_Q(X/zX) \longrightarrow X \longrightarrow 0.$$

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Pf: Pull back:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & P \arrow[r] \arrow[d] & X \arrow[d, "z"] & \\
0 \arrow[r] & \Omega X \arrow[r] & F \arrow[r] & X \arrow[r] \arrow[d] & 0 \\
 & & & \bar{X} \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

$\Rightarrow P = \Omega \bar{X}$ and the top row is what we want. ✓.

Cor Let $R = S/(f)$, $R^\# = S[y]/(f+y^n)$. Let N be a MCM $R^\#$-module. Then for $k = 1, \_\_, n$ we have

$$✪ \quad 0 \to \Omega_{R^\#}(N) \to \Omega_{R^\#}(N/y^kN) \to N \to 0$$

The sequence split iff $y^k \operatorname{Ext}^1_{R^\#}(N, \Omega_{R^\#}N) = 0$

FACT. ny^{n-1} always kills that Ext, so we recover Herzog-Popescu.

Define subcategories $\Sigma_k \subseteq \mathrm{MCM}(R^\#)$

$$\Sigma_k = \operatorname{add}\{\Omega_{R^\#}(M) \mid M \text{ MCM } / S[y]/(f,y^k)$$

Then $\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots \subseteq \Sigma_{n-1} \underset{\uparrow \text{H-P.}}{=} \mathrm{MCM}\, R^\#$

Thm [Dugas-GL] For any $N \in \mathrm{MCM}\, R^\#$, the SES $\circledast$ is a Σ_k-approximation of N.

I.e. any homom $\Omega_{R^\#}(M) \longrightarrow N$, where M is MCM over $S[y]/(f, y^k)$, factors through the map
$$\Omega_{R^\#}(N/y^k N) \longrightarrow N.$$

Application to hypersurfaces of FRT.
Let $R = S/(f)$ be an ADE (simple) hypersurface.

Let G be a representation generator, so $\mathrm{MCM}\, R = \mathrm{add}\, G$.

Thm [Iyama, GL, Quarles 2008]
The endomorphism ring
$$\Lambda = \mathrm{End}_R(G)$$
has finite global dimension.

(Behaves somewhat like a res. of. sings.)

Now let $R^\# = S[y]/(f + y^n)$, $n \geqslant 2$.

If $n > 2$, then $R^\#$ does not have FRT.

$$\begin{aligned}\underline{\text{However}}\quad \Sigma_1 &= \operatorname{add}\{\Omega_{R^\#}(M) \mid M \text{ MCM over } S[y]/(f, y^1)\}\\ &= \operatorname{add}\{\Omega_{R^\#}(M) \mid M \in \operatorname{MCM} R\}\\ &= \operatorname{add}\{\Omega_{R^\#}(G)\}.\end{aligned}$$

This does contain only finitely many indecomposables.

Set

$$\Lambda^\# = \operatorname{End}_{R^\#}(\Omega_{R^\#}(G)).$$

<u>Thm</u> [D-L] $\Lambda^\#$ is a "non-commutative hypersurface", in the sense that every fin. gen. $\Lambda^\#$-module has a proj. resolution which is eventually 2-periodic. ("NC MF").

<u>Proof</u>. It suffices to prove that when N is a MCM $R^\#$-module, the $\Lambda^\#$-module

$$\operatorname{Hom}_{R^\#}(\Omega_{R^\#}(G), N)$$

has a 2-periodic resolution.

We have the Σ_1-approximation ⊛:

$$0 \to \Omega_{R^\#}(N) \to \Omega_{R^\#}(N/yN) \to N \to 0$$

and a similar one with N and $\Omega_{R^\#}(N)$ switched.

Since they are Σ_1-approximations and $\Sigma_1 = \operatorname{add} \Omega_{R^\#}(G)$, applying $\operatorname{Hom}_{R^\#}(\Omega_{R^\#}(G), -)$ to them leaves them exact.

They can then be spliced together to give a long 2-periodic exact seq

$$\cdots \longrightarrow (\Omega_\#(G), \Omega_\#(N/yN)) \longrightarrow (\Omega_\#(G), \Omega_\#(\Omega_\# N / y\Omega_\# N)) \cdots$$

Now $N/yN \in \operatorname{add} G$, so $\Omega_\#(N/yN) \in \Sigma_1$ and each of these is projective over $\Lambda^* = \operatorname{End}_{R^\#}(\Omega_\#(G))$. ✓ □