What is a non-commutative desingularization?

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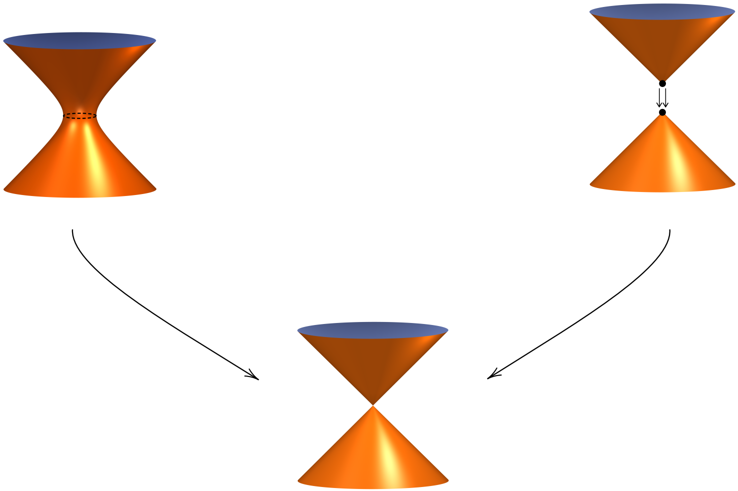


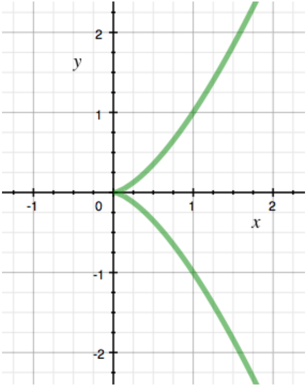
What is a Resolution of Singularities?

Suppose X is an algebraic variety, meaning the solution set of a system of polynomial equations. In general, X might have singularities, or non-smooth points, which prevent it from being a manifold.

Definition (Ver. 1)

A resolution of singularities of X is a parametrization of X by the points of a manifold.

Example: the cusp



Consider the cusp $C\colon \{x^3 = y^2\}$.

We can parametrize C by the points of a line: $t \mapsto (t^2, t^3)$.

Not only is this a parametrization, it's a smooth map (almost) everywhere, in the sense that the derivatives of the coordinate functions don't simultaneously vanish (except at the origin).

Another way to think of resolving this singularity:

This is the parametrization $t \mapsto (t^2, t^3, t)$. (This corresponds to the "blowup algebras" coming later.) Now the derivatives really don't simultaneously vanish.

More precisely...

Definition (Ver. 2)

A resolution of singularities is a map of varieties $\pi\colon \widetilde{X} \longrightarrow X$ where

- $\widetilde{X}$ is a manifold;
- π is a surjective map;
- π is differentiable, and almost everywhere a diffeomorphism;
- π is *proper*, which over $\mathbb{C}$ means that π^{-1}(compact) is compact.

In general, properness is a kind of finite generation assumption.

Why resolve singularities?

Nonsingular varieties are *much* easier to work with than singular ones, and the definition is built to transmit the information from the resolution to the resolved variety.

Many technical results in algebraic geometry, for example on vanishing of cohomology (Kodaira Vanishing, Kawamata-Viehweg Vanishing, . . .) are proved in exactly this way.

Other examples come from differential equations, dynamical systems, etc.

Definition (Ver. 3, Algebra (preliminary))

Let R be an integral domain with quotient field Q. A resolution of singularities of R is an intermediate ring $R \subseteq A \subseteq Q$ such that A is regular and a finitely generated R-module.

The condition that A is regular means precisely that A corresponds to a smooth variety. Generally speaking, we can think of A as a polynomial (or power series) ring over $\mathbb{C}$.

An extension of domains $R \subseteq A$ that share the same quotient field is called *birational*. Geometrically, it means that the corresponding varieties have the same field of rational functions.

The cusp again

The cusp C corresponds to the ring $R = \mathbb{C}[x, y]/(x^3 - y^2)$. Using the parametrization we've already seen, this is isomorphic to $\mathbb{C}[t^2, t^3]$ with quotient field $\mathbb{C}(t)$, the full field of rational functions. We can thus take

$$A = \mathbb{C}[t] .$$

In this case, all we did was "normalize". More on that later.

Whitney Umbrella

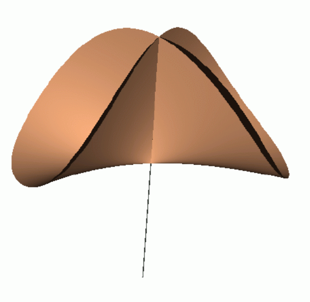
Let $W : \{x^2 = y^2 z\}$, or equivalently

$$R = \mathbb{C}[x, y, z]/(x^2 - y^2 z)\,.$$

This time it's much less obvious (though easy to check) that we can parametrize W by

$$(s, t) \mapsto (st, t, s^2)\,.$$

So a resolution of singularities of $R \cong \mathbb{C}[st, t, s^2]$ is $\mathbb{C}[s, t] \cong R[\frac{x}{y}]$.



An important change to the algebraic definition

In general, the requirement that $A \subseteq Q$ is too strong. We need to allow gluing together of regular birational extensions, in the same way manifolds are glued together from Euclidean patches.
On the algebraic side, the gluing leads to a blowup algebra

$$A = R[It]$$

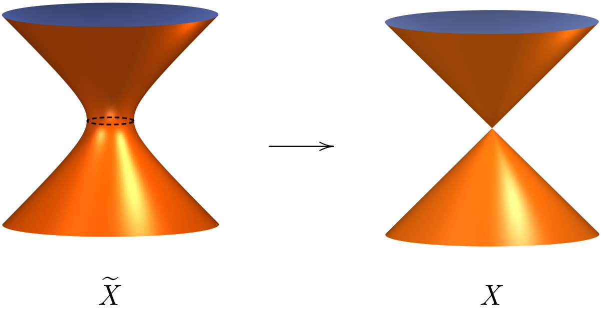
for an ideal I of R. It is longer birational. In particular, it is almost never finitely generated as an R-module.

A blowup algebra should be considered smooth if its constituent pieces are — *but* $R[It]$ *may not be a polynomial ring itself.* Instead, it is smooth "on the punctured spectrum."

Example: The Cone

Let $C\colon \{z^2 = x^2 + y^2\}$, equivalently $R = \mathbb{C}[x, y, z]/(z^2 - x^2 - y^2)$.

There is a standard geometric resolution of singularities of C obtained by blowing up the vertex, replacing it by a $\mathbb{P}^1_{\mathbb{C}}$.

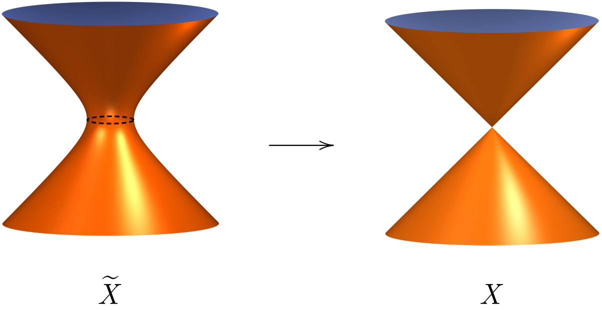


Example: The Cone

Algebraically, blowing up the vertex defines the blowup algebra

$$A = R[xt, yt, zt] \subset R[t]$$
$$\cong \mathbb{C}[x, y, z, a, b, c]/(z^2 - x^2 - y^2,\ ay - bx,\ az - cx,\ bz - xy)$$

then analyzing two charts, $a \neq 0$ and $b \neq 0$.



Can it be done? Yes, *usually*

Curves

Normalization (a.k.a. integral closure) resolves singularities of curves. Known to Italian geometers in the 19th century; proven carefully by (at least) Kronecker and Max Noether.

In this case no blowup algebras are needed; one really can find $A \subseteq Q$.

Surfaces defined over $\mathbb{C}$

Walker (1936) and Zariski (1939). Now blowup algebras are really required.

3-folds defined over $\mathbb{C}$

Zariski (1944)

Surfaces or 3-folds in char. p

Abhyankar (1956, 1966)

Can it be done?

Any variety over $\mathbb{C}$

Hironaka (1964)

The proof is 200 pages, and consists (among other things) of 14 nested inductions. It's been called the most complicated mathematical work in history.

The proof has been streamlined somewhat since then, notably by Villamayor, Bierstone–Milman, Encinas–Hauser. It's also been made constructive (Benito–Encinas–Villamayor), which the original proof was not. Still, it remains difficult.

Some petty complaints

Constructing a resolution of singularities of a given variety ...

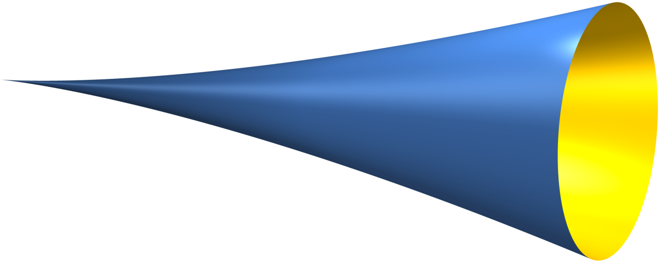
- is generally very complicated.
- uses a great deal of geometric machinery.
- introduces blowup algebras and their punctured spectra, which are difficult to analyze.
- can't be done (yet) over fields of prime characteristic or the integers.

Naive hope

Is there a construction that has the same applications, but . . .

- avoids the blowup algebras, possibly sticking to module-finite R-algebras?
- works over any ground ring?

No. For example, the cone $\mathbb{C}[x, y, z]/(z^2 - x^2 - y^2)$ has no regular finite extensions inside its quotient field. Neither does the trumpet $z^2 = x^2 + y^3$.



A potential solution: allow non-commutative rings A

The idea seems first to have arisen in theoretical physics (?!).

Resolution of Stringy Singularities by Non-commutative Algebras (Berenstein-Leigh '01)

"Our intention in this paper is to make a general proposal for 'resolving' singularities within non-commutative geometry and to understand the D-branes on these spaces. [...] The algebraic geometry of this commutative algebra will be identified with the target space where closed strings propagate, and the algebraic geometry of the non-commutative algebras will give the resolution of those singularities. In this sense a commutative singularity can be made smooth in a non-commutative sense."

A potential solution: allow non-commutative rings A

This also forces a change in philosophy: we'll focus on *homological aspects of the modules* over A, rather than ring-theoretic properties.

In particular, we'll think of two rings as "the same" if they have equivalent module categories, specifically if they're Morita equivalent:

$$A \simeq_{\text{Morita}} B\,.$$

In particular this holds if

$$B = \operatorname{End}_A(A^n) \cong \operatorname{Mat}_n(A)$$

for some free A-module A^n.

Birationality

Problem

It makes no sense to insist that $A \subseteq Q$ anymore, or even that A is gotten by gluing together such rings, as that would force A to be commutative.

Solution

We force A to sit inside some ring Morita equivalent to Q. Since Q is a field, this means

$$A \subseteq \mathrm{Mat}_n(Q)$$

for some n.

We also now insist A is finitely generated over R.

Regularity

Question

What does "regular" or "smooth" mean for non-commutative rings?

Solution

Translating into homological algebra, this should mean finite global dimension, every module having finite projective dimension.

Properness

Question

What should replace the "properness" assumption?

Solution

We assume that A is a finitely generated R-module.

A candidate definition

So in total, we seek R-algebras A which

- are finitely generated R-modules;
- have finite global dimension; and
- sit inside $\mathrm{Mat}_n(Q)$ for some n.

Problem

Finite global dimension is much weaker for non-commutative rings than for commutative ones. We'll need to impose extra conditions to have a functional theory.

In particular, we should at least assume A is homologically homogeneous: every simple A-module has the same projective dimension. Even this isn't quite enough.

Crepant resolutions

The most hospitable resolutions of singularities are the crepant ones. A resolution $\pi\colon \widetilde{X} \longrightarrow X$ is crepant if the *canonical bundles* are isomorphic:

$$\pi^*\omega_X = \omega_{\widetilde{X}}\,.$$

This has good technical implications for the transmission of information from $\widetilde{X}$ to X.

Existence of a crepant resolution is quite strong: among other things, it forces X to have rational singularities, and the corresponding ring R to be a Gorenstein normal domain.

Finally, the definition

Definition (Van den Bergh)

Let R be a Gorenstein normal domain. A non-commutative crepant resolution of singularities of R is a homologically homogeneous ring Λ of the form

$$\Lambda = \mathrm{End}_R(M)$$

for M a finitely generated reflexive R-module.

Equivalently, $\Lambda = \mathrm{End}_R(M)$ is a non-commutative crepant resolution if

- Λ has finite global dimension;
- Λ is maximal Cohen–Macaulay as an R-module.

Two justifying theorems

Theorem (Van den Bergh, '04)

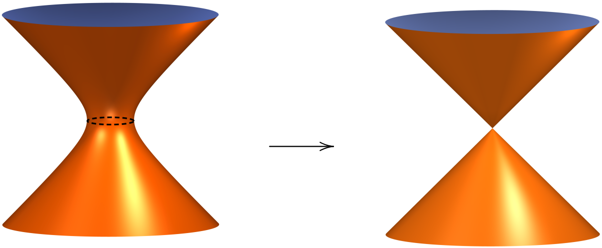
Assume that $\dim R = 3$ *and* $\operatorname{Spec} R$ *has terminal singularities. Then* R *has a non-commutative crepant resolution if and only if it has a commutative crepant resolution.*

Theorem (Stafford-Van den Bergh, '06)

If R *has characteristic zero and possesses a non-commutative crepant resolution of singularities, then it has (at worst) rational singularities.*

Example: The Cone, revisited

Let $C\colon \{z^2 = x^2 + y^2\}$, equivalently $R = \mathbb{C}[x,y,z]/(z^2 - x^2 - y^2)$.



$$\widetilde{X} \longrightarrow X$$

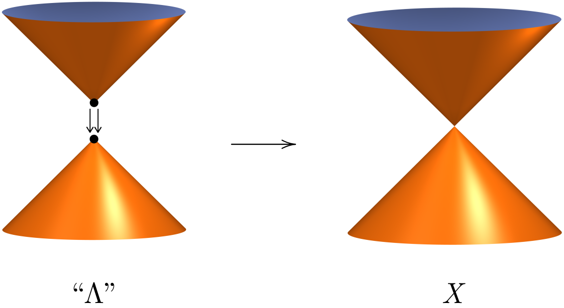
Example: The Cone, revisited

Let I be the ideal (x, z), and set $\Lambda = \mathrm{End}_R(R \oplus I) \cong \begin{pmatrix} R & I \\ I^{-1} & R \end{pmatrix}$.

One can show that there is an equivalence of derived categories

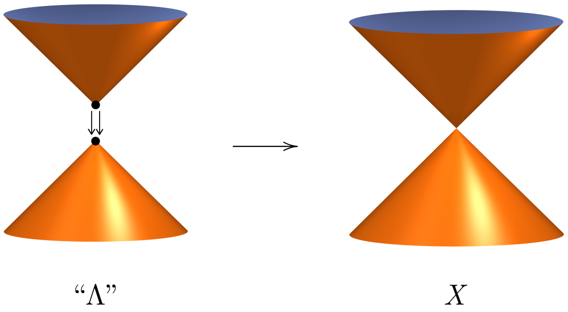
$$D^b(\mathrm{Mod}\,\Lambda) \simeq D^b(\mathrm{Qcoh}\,\widetilde{X})\,,$$

so from a derived-categorical point of view, Λ "is" the resolution $\widetilde{X}$.



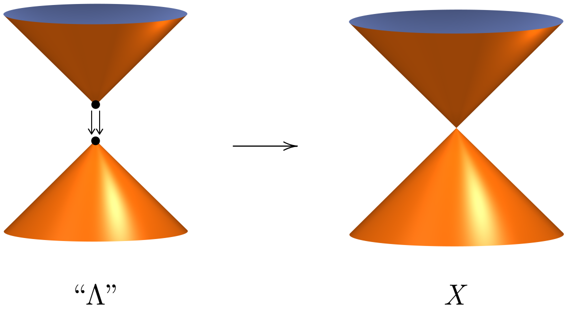
Example: The Cone, revisited

Algebraically, the points of X correspond to simple modules over $R = \mathbb{C}[x,y,z]/(z^2 - x^2 - y^2)$. Non-isomorphic simple Λ-modules are non-isomorphic when considered as R-modules, except for a 1-parameter family.



Example: The Cone, revisited

The simple modules in that family are non-zero representations of the quiver $\bullet \rightrightarrows \bullet$ with dimension vector $(1, 1)$. There is exactly a $\mathbb{P}^1$ of these.



Generic determinantal rings

Theorem (Buchweitz-Leuschke-Van den Bergh)

Let $X = (x_{ij})$ be the generic $m \times n$ matrix, $S = K[\{x_{ij}\}]$, and $R = S/I_t(X)$ the generic determinantal ring defined by the vanishing of the $t \times t$ minors. Then there is an R-module M such that $\Lambda = \mathrm{End}_R(M)$ has finite global dimension and is maximal Cohen-Macaulay as an R-module.

If $m = n$ (so that R is Gorenstein) then Λ is a non-commutative crepant resolution.

Construction in the case of maximal minors ($t = m$)

Consider $X = (x_{ij})_{m \times n}$ as a map between two free S-modules of rank $n \geq m$:

$$0 \longrightarrow \mathcal{G} \xrightarrow{X} \mathcal{F} \longrightarrow M_1 = \operatorname{coker} X \longrightarrow 0\,.$$

Take exterior powers:

$$0 \longrightarrow \bigwedge^k \mathcal{G} \xrightarrow{\bigwedge^k X} \bigwedge^k \mathcal{F} \longrightarrow M_k \longrightarrow 0$$

for $k = 1, \ldots, m$ (in particular $M_m \cong S/I_m(X) = R$).

Fact

Each M_k is a maximal Cohen–Macaulay R-module of rank $\binom{m}{k}$.

This is essentially a rephrasing of Cramer's Rule (!).

Maximal minors, precise version

Theorem

Set

$$M = \bigoplus_{k=1}^{m} M_k = \bigoplus_{k=1}^{m} \operatorname{coker}\left(\bigwedge^{k} X\right).$$

and $\Lambda = \operatorname{End}_R(M)$*. Then* Λ *is MCM over* R *and has finite global dimension.*

Idea of the Proof, maximal minors case

The determinantal variety defined by the vanishing of the maximal minors of X has a natural resolution of singularities.

Think of $\operatorname{Spec} S = K^{m \times n}$ as the affine space of $m \times n$ matrices over K. Then $\operatorname{Spec} R \subset \operatorname{Spec} S$ is the subset of matrices of rank $< m$.

Consider the incidence variety

$$\mathcal{Z} = \left\{ (H, \theta) \;\middle|\; \begin{array}{l} H \text{ hyperplane in } K^m \\ \theta \colon K^n \longrightarrow K^m \\ \operatorname{image} \theta \subset H \,. \end{array} \right\}$$

Since θ has rank $< m$ iff its image is contained in a hyperplane, the projection $(H, \theta) \mapsto \theta \in \operatorname{Spec} S$ actually lands in $\operatorname{Spec} R$.

$\mathcal{Z}$ is called the Springer resolution of singularities of R.

Idea of the Proof, maximal minors case

The Springer desingularization comes equipped with projections to $\operatorname{Spec} R$ and $\mathbb{P} = \mathbb{P}^{n-1}$:

$$\begin{tikzcd}
\mathcal{Z} \arrow[d, "q"'] \arrow[dr, hook] \arrow[drr, bend left, "p"] & & \\
 & \mathbb{P} \times \operatorname{Spec} S \arrow[r] \arrow[d] & \mathbb{P} \\
\operatorname{Spec} R \arrow[r, hook] & \operatorname{Spec} S &
\end{tikzcd}$$

Plan: Identify the R-modules M_k as "coming from" $\mathbb{P}$. Find coherent sheaves $\mathcal{L}_k$ on $\mathbb{P}$ so that $M_k = q_* p^* \mathcal{L}_k$.

Idea of the Proof, maximal minors case

In fact the $\mathcal{L}_k$ are the sheaves of order-$(k-1)$ differential forms.

Proposition

$M_k = \mathcal{R}q_* p^*(\Omega_{\mathbb{P}}^{k-1}(k))$, that is, the direct image of $p^*\Omega^{k-1}(k)$ is M_k and the higher direct images vanish.

This allows us to get an explicit S-free resolution for each $\mathrm{Hom}_R(M_i, M_j)$ and observe that they are MCM. The vanishing further implies the equivalence of derived categories between Λ and $\mathcal{Z}$, which forces Λ to have finite global dimension.

Arbitrary Minors

To prove the general case:

- $R = K[X]/I_t(X)$ for some $1 \leq t \leq m$.
- Instead of exterior powers of X, consider all Schur functors S_λ such that λ is a partition fitting in a $\times(m-t)$ box.
- Replace hyperplanes $H \subset K^m$ with $(t-1)$-planes $L \in \mathrm{Grass}(t-1, K^m)$.
- Replace the sheaves of differential forms with $S_\lambda \mathcal{Q}$, where $\mathcal{Q}$ is the tautological quotient bundle of $\mathrm{Grass}(t-1, K^m)$.

Then the direct sum of $q_* p^* S_\lambda \mathcal{Q}$ is the R-module M, and its endomorphism ring is a MCM module of finite global dimension. If $m = n$, it is a non-commutative crepant resolution.

Thank you!