A generalization of Knörrer periodicity

joint work with A. Dugas

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Setup

Throughout,

- $S = k[\![x_0, \ldots, x_d]\!]$ is a complete regular local ring of dimension $d+1$.
- $f \in S$ is a nonzero nonunit.
- $R = S/f$ is the hypersurface ring.

- $R^{\#} = S[y]/(f + y^n)$ for some $n \geq 2$ is the n-fold branched cover.

We have functors between R-modules and $R^{\#}$-modules:

$$M \longmapsto \Omega_{R^{\#}}(M)$$
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Assume $n = 2$ and $\operatorname{char} k \neq 2$.

Theorem (Knörrer)

For any MCM module M over R, we have

$$\Omega_{R\#}(M)/y\Omega_{R\#}(M) \cong M \oplus \Omega_R(M)\,.$$

For any MCM module N over $R^{\#}$, we have

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Consequence: classification of hypersurfaces of finite CM type

Theorem (Buchweitz-Greuel-Schreyer-Knörrer 1987)

Let $R = k[\![x_0, \ldots, x_d]\!]/(f)$, where k is an alg. closed field of characteristic $\neq 2, 3, 5$. Then R has finite MCM type if and only if R is isomorphic to the hypersurface defined by

$$g(x_0, x_1) + x_2^2 + \cdots + x_d^2\,,$$

where $g(x_0, x_1)$ is one of the following polynomials.

$$\begin{array}{rl} (A_n) & x_0^2 + x_1^{n+1} \\ (D_n) & x_0^2 x_1 + x_1^{n-1} \\ (E_6) & x_0^3 + x_1^4 \\ (E_7) & x_0^3 + x_0 x_1^3 \\ (E_8) & x_0^3 + x_1^5 \end{array}$$

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Generalization to $f + y^n$

Now let $n \geq 2$ be aribitrary (but relatively prime to $\operatorname{char} k$).

Theorem (Herzog-Popescu, 1997)

For any MCM module N over $R^{\#}$, we have

$$\Omega_{R^{\#}}(N/y^{n-1}N) \cong N \oplus \Omega_{R^{\#}}(N)\,.$$

Notice that $N/y^{n-1}N$ is not an R-module, but is a module over the complete intersection ring $S[y]/(f, y^{n-1})$.

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Key Lemma

Lemma

Let Q be a commutative Noetherian ring and X a finitely generated module. Let $z \in Q$ be a non-zerodivisor on X. Then there is a short exact sequence

$$0 \longrightarrow \Omega_Q(X) \longrightarrow \Omega_Q(X/zX) \longrightarrow X \longrightarrow 0$$

of Q-modules.

Proof.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_Q(X) \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "z"] & 0 \\
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The short exact sequences

Corollary

Let $R = S/(f)$ and $R^{\#} = S[y]/(f + y^n)$, $n \geq 2$. Let N be a MCM $R^{\#}$-module. Then for every $k = 1, \ldots, n$ there is a short exact sequence

$$0 \longrightarrow \Omega_{R^{\#}}(N) \longrightarrow \Omega_{R^{\#}}(N/y^k N) \longrightarrow N \longrightarrow 0 \,.$$

The sequence is split exact if and only if y^k annihilates $\mathrm{Ext}^1_{R^{\#}}(N, \Omega_{R^{\#}}(N))$.

In particular, ny^{n-1} always annihilates, so if $n \not| \operatorname{char} k$ then the sequence with $k = n - 1$ is split exact, recovering the result of Herzog-Popescu.

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Approximations

Definition

Let $\mathcal{A}$ be an abelian category, $\mathcal{X} \subseteq \mathcal{A}$ a full subcategory, and M an object of $\mathcal{A}$.

(i) A *right $\mathcal{X}$-approximation* (or *$\mathcal{X}$-precover*) of M is a morphism $q\colon X \longrightarrow M$, with $X \in \mathcal{X}$, such that any $f\colon X' \longrightarrow M$ with $X' \in \mathcal{X}$ factors through q. Equivalently, the induced homomorphism $\mathrm{Hom}_{\mathcal{A}}(X', X) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X', M)$ is surjective.

(ii) A right $\mathcal{X}$-approximation $q\colon X \longrightarrow M$ is *minimal* (or a *$\mathcal{X}$-cover*) if every endomorphism $\varphi \in \mathrm{End}_{\mathcal{A}}(X)$ with $\varphi q = q$ is an isomorphism.

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Approximations by syzygies

We have for any MCM $R^{\#}$-module and any $k = 1, \ldots, n$ the short exact sequence

$$0 \longrightarrow \Omega_{R\#}(N) \longrightarrow \Omega_{R\#}(N/y^k N) \longrightarrow N \longrightarrow 0\,.$$

Main Theorem

Let Σ_k denote the full subcategory of $R^{\#}$-modules which are of the form $\Omega_{R\#}(M)$, where M is a MCM module over $S[y]/(f, y^k)$.

The short exact sequence above is a right Σ_k-approximation of N and a left Σ_k-approximation of $\Omega_{R\#}(N)$.

In other words, any homomorphism $\Omega_{R\#}(M) \longrightarrow N$, where M is killed by f and y^k, factors through $\Omega_{R\#}(N/y^k N)$.

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Application to simple singularities

Here is an application to hypersurface rings of finite MCM representation type (simple singularities, ADE singularities).

Assume $R = S/(f)$ is an ADE singularity. Let G be a representation generator for R, so that every indecomposable MCM R-module is isomorphic to a direct summand of G.

Theorem (Iyama, Leuschke, Quarles)

The endomorphism ring $\Lambda = \operatorname{End}_R(G)$ has finite global dimension at most $\max\{2, d\}$, with equality holding if $d \geq 2$.

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Now let $R^{\#} = S[y]/(f + y^n)$, $n \geq 2$.

Recall that Σ_1 is the full subcategory of $R^{\#}$-modules of the form $\Omega_{R^{\#}}(M)$, where M is killed by f and y; that means M is a MCM R-module.

Since R has finite CM type, Σ_1 contains only finitely many indecomposables, namely, all summands of $\Omega_{R^{\#}}(G)$.

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Let $R = S/f$ be a simple singularity of dimension d, with MCM generator G, let $R^{\#} = S[y]/(f + y^n)$, and let

$$\Lambda^{\#} = \mathrm{End}_{R^{\#}}(\Omega_{R^{\#}}(G)) \,.$$

Then every finitely generated $\Lambda^{\#}$-module has a projective resolution which is periodic of period at most 2 after at most d steps.

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Proof.

It suffices to show that any $\Lambda^{\#}$-module of the form $\mathrm{Hom}_{R^{\#}}(\Omega_{R^{\#}}(G), N)$, where N is a MCM $R^{\#}$-module, has a periodic resolution.

Since they are Σ_1-approximations, the two exact sequences

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$$0 \longrightarrow N \longrightarrow \Omega_{R^{\#}}(\Omega_{R^{\#}}(N)/y\Omega_{R^{\#}}(N)) \longrightarrow \Omega_{R^{\#}}(N) \longrightarrow 0$$

remain exact after applying $\mathrm{Hom}_{R^{\#}}(\Omega_{R^{\#}}(G), -)$.

Splice them together to get a $\Lambda^{\#}$-projective resolution of $\mathrm{Hom}_{R^{\#}}(\Omega_{R^{\#}}(G), N)$. □

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