CONNECTIONS BETWEEN NONCOMMUTATIVE GEOMETRY AND COMMUTATIVE ALGEBRA VIA NONCOMMUTATIVE RESOLUTIONS

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Abstract. These lectures will describe the work done over the last several years, by many sets of authors, on non-commutative analogues of resolutions of singularities. I will start by discussing the McKay Correspondence, a key motivating example, and then consider several desirable features of any potential definition of the phrase "non-commutative desingularization", including Van den Berghs definition of non-commutative crepant resolutions. Throughout I will emphasize the role played by maximal Cohen-Macaulay modules.

Contents

Many thanks to the organizers for the honor of giving this series of 3 lectures.

My goal in these lectures is to explain and motivate the notion of "non-commutative resolution of singularities". I will emphasize the role of maximal Cohen-Macaulay modules over Cohen-Macaulay rings.

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Part 1. What should a non-commutative desingularization be?

The aim of this lecture is to discuss what properties a ring homomorphism $R \longrightarrow \Lambda$, where R is a commutative ring and Λ is not necessarily commutative (but R maps into the center $Z(\Lambda)$) should have in order to best mimic the properties of a resolution of singularities $\widetilde{X} \longrightarrow X$ of an algebraic variety X.

Guiding Principle: We should adopt a representation-theoretic point of view, that is, we should view module categories, or perhaps derived categories of modules, as our main objects of consideration, rather than the concrete ring-theoretic properties. In particular, for any ring Λ, we have $\operatorname{mod}\Lambda \simeq \operatorname{mod} M_n(\Lambda)$ by Morita equivalence. We take this to mean that $\operatorname{mod}(-)$ "can't tell Morita-equivalent rings apart", so in particular it's blind to commutativity.

Like any guiding principle, strict adherence leads to monastic aridity. If one takes the GP as a Commandment, one ends up with "categorical resolutions of singularities" [Bondal-Orlov, Kuznetsov-Lunts]. These are too abstract for me, so I will insist that our categories are honest module categories.

There is a general definition of categorical desingularization:

Definition (Bondal-Orlov). A categorical desingularization of a triangulated category D is an abelian category C of finite global dimension, and a triangulated subcategory $K \subseteq D^b(C)$, closed under direct summands, such that

$$D \simeq D^b(C)/K\,.$$

This is too general to do much with, though it does have the advantage of always existing (given Hironaka):

Theorem (Kuznetsov-Lunts). *The derived category of any singularity over a field of characteristic 0 has a categorical resolution in this sense.*

So what properties of resolutions of singularities are we attempting to mimic?

Recall that a usual ("geometric") resolution of singularities of a variety (scheme) X is a map $\pi\colon \widetilde{X} \longrightarrow X$ such that

(1) $\widetilde{X}$ is non-singular (smooth);
(2) π is proper (over $\mathbb{C}$, π^{-1}(compact) is compact);
(3) π is birational (induces isomorphism on function fields).

What would qualify $R \longrightarrow \Lambda$ to count as a resolution of singularities of R?

(iii) $\pi\colon \widetilde{X} \longrightarrow X$ is birational. This essentially means that π induces an isomorphism on quotient fields. Morally, this means

$$\mathcal{O}_{\widetilde{X}} \otimes_{\mathcal{O}_X} K = K$$

where K is the quotient field of $\mathcal{O}_X$.

Since we only care about module categories, we should weaken this equality by replacing it with "Morita equivalent to" (or even "derived equivalent to"). But the only rings Morita equivalent to a field K are the matrix rings $M_n(K)$. So we ask

$$\boxed{\Lambda \otimes_{\mathcal{O}_X} K \cong M_n(K)\,.}$$

Call Λ birational (over R) if this is true.

(ii) π is proper. If we expect to have the boxed isomorphism above, then we should ask that

$$\boxed{\Lambda \text{ is a finitely generated } R\text{-module.}}$$

Since finite extensions are proper, this is at least as strong as (ii). Also, it is known that proper maps between affine schemes are finite, so this is reasonable.

(i) $\widetilde{X}$ is smooth. Clearly we should (at least) insist that

$$\boxed{\Lambda \text{ has finite global dimension.}}$$

So we arrive at a first, very weak, definition.

Definition: Let R be a commutative Noetherian ring. A useless non-commutative resolution of R (or of $\operatorname{Spec} R$) is a module-finite algebra Λ which is birational and has finite global dimension.

First we note that even this extremely weak definition rules out the possibility that we might be able to take Λ to be commutative. There are easy examples of rings that have no regular finite extensions inside their quotient fields, e.g. $\mathbb{C}[x,y,z]/(z^2 - x^2 - y^2)$. So we really do need to talk about non-commutative rings.

However, this definition is much too weak. The main problem is that finite global dimension is a very weak, and badly-behaved, property for non-commutative rings. Among other problems,

- there is no analog of the Auslander-Buchsbaum formula, which in particular says that if M has finite projective dimension then $\operatorname{pd}_R(M)$ is bounded by the (Krull) dimension.
- there are lots of pathological examples (e.g. a Noetherian local ring of global dimension 3 with no (non-trivial) quotients of finite global dimension (so no induction possible)
- finite global dimension does not imply any Gorenstein or Cohen-Macaulay property as in the commutative case.
- it does not localize well.

We will solve all of these by adding in three hypotheses.

The last issue is easy to address:

1

Definition. An R-algebra Λ is non-singular if $\operatorname{gldim} \Lambda_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$.

This is inconvenient to check in practice, so we will be on the lookout for situations when finite global dimension implies non-singularity.

2

Now we add in a Cohen-Macaulayness assumption. From now on, R will be local.

Definition. An R-algebra Λ is an R-order if it is maximal Cohen-Macaulay as R-module, that is $\operatorname{depth}_R \Lambda = \dim R$.

More about MCM modules in a little while.

Facts about non-singular R-orders: Let R be commutative Noetherian local.

- Λ is a non-singular R-order if and only if Λ is homologically homogeneous, meaning that all its simple modules have the same (finite) projective dimension.
- [Iyama-Wemyss] If R is a k-algebra and Λ is a non-singular R-order with a finite group G such that $|G| \in k^\times$ acting on Λ, then the skew group ring $\Lambda \# G$ is also a non-singular R-order. (McKay correspondence)

Finally we will impose a Gorenstein-ness assumption. For this it is convenient to restrict our base ring R to be Cohen-Macaulay from now on. This is relatively mild, and still includes many examples of interest, such as hypersurface rings, complete intersections, rings of invariants, determinantal rings. . . .

3

Definition. Let R be a CM local ring with canonical module ω. Let Λ be an R-order. Say that Λ is a Gorenstein order if $\operatorname{Hom}_R(\Lambda, \omega)$ is a projective Λ-module. (Only on one side.)

Facts about Gorenstein R-orders: Let R be CM and let Λ be an R-order.

- Λ is non-singular iff $\operatorname{gldim} \Lambda < \infty$ and Λ is Gorenstein. (So we don't have to check the global dimension at every prime.)
- If Λ is Gorenstein then it satisfies the Auslander-Buchsbaum formula for all finitely generated modules of finite projective dimension.

Here is a stronger property which will also be useful.

Definition. Let R, Λ be as above. Say that Λ is a symmetric R-algebra if

$$\operatorname{Hom}_R(\Lambda, R) \cong \Lambda$$

(as a two-sided Λ-module) via the map that sends $f\colon M \longrightarrow M$ to $\operatorname{trace}(?f) \in \operatorname{Hom}_R(\Lambda,\ R)$.

If R is Gorenstein (so that $\omega \cong R$), then Λ symmetric implies Λ Gorenstein.

Turns out that the symmetry condition is actually equivalent to Λ being an endomorphism ring, in the following sense. First of all, note that endomorphism rings are symmetric algebras:

Proposition (Auslander). *If M is a reflexive R-module then $\Lambda = \operatorname{End}_R(M)$ is a symmetric R-algebra.*

For the other direction, recall from classical commutative algebra that if R is a domain with quotient field K, then:

- A classical order over R is a module-finite algebra Λ contained in a simple K-algebra D, with $\Lambda \otimes_R K = D$.
- Such a thing is a maximal order if it is maximal under inclusion.

Maximal orders are the best-behaved ones, and are well-studied over, for instance, Dedekind domains.

Proposition. *Let R be a normal domain with quotient field K. Let Λ be a symmetric, non-singular R-order contained in $M_n(K)$ and such that $\Lambda \otimes_R K = M_n(K)$. Then Λ is a maximal order in $M_n(K)$.*

Proof. Since Λ is reflexive over R, you can check maximality at the height-one primes. So assume that R is a DVR. Then Λ, being non-singular, is a hereditary ring, so is Morita equivalent to a product of rings $T_m(\Delta)$, where Δ is the unique maximal order in $M_n(K)$ and $T_m(K)$ is the ring of matrices $(a_{ij}) \in M_n(\Delta)$ with $a_{ij} \in \operatorname{rad}\Delta$ for all $i > j$.

But such a product of rings is symmetric only if $m = 1$ and there is only one factor. So $\Lambda = \Delta$. $\square$

Here is the point.

Theorem (Auslander-Goldman)**.** *Let R be a normal domain, K its quotient field. If Λ is a maximal order in $M_n(K)$, then $\Lambda \cong \operatorname{End}_R(M)$ for some reflexive R-module M.*

Proof. Choose any $v \neq 0$ in K^n, and define $\varphi\colon M_n(K) \twoheadrightarrow K^n$ by evaluation at v. Set $E_1 = \varphi(\Lambda)$. Then E_1 is a finitely generated R-module and $\operatorname{End}_R(E_1)$ is an order in $M_n(K)$ containing Λ. Replacing E_1 by $E = E_1^{**} \subset K^n$ makes the endomorphism ring larger, if anything, but by maximality $\operatorname{End}_R(E) = \Lambda$. $\square$

Corollary. *The following are equivalent for a module-finite algebra Λ over a Gorenstein local ring R:*

(1) Λ is a Gorenstein ($\iff$ symmetric, since R is Gorenstein) birational non-singular R-order.
(2) $\Lambda \cong \operatorname{End}_R(M)$ for some reflexive R-module, Λ is MCM as an R-module, and $\operatorname{gldim}\Lambda < \infty$.
(3) $\Lambda \cong \operatorname{End}_R(M)$ as above, and is homologically homogeneous.

These are the best substitutes for resolution of singularities found so far.

So finally we arrive at Van den Bergh's definition.

Definition (Van den Bergh)**.** Let R be a Gorenstein normal domain. A non-commutative crepant resolution of R is an algebra $\Lambda = \mathrm{End}_R(M)$, where M is a reflexive R-module, such that Λ has finite global dimension and is MCM as R-module.
Equivalently, Λ is a symmetric birational non-singular R-order.

Question: What is the (a?) correct definition if R is not assumed to be Gorenstein?

Here is one answer.

Definition (Iyama-Wemyss if R is CM, Dao-Faber-Ingalls in general)**.** Let R be a Noetherian commutative ring. A non-commutative crepant resolution (NCCR) of R is an algebra $\Lambda = \mathrm{End}_R(M)$, where M is a torsion-free module with full support ($\mathrm{Supp}\, M = \mathrm{Spec}\, R$) and such that Λ is a non-singular R-order.

(Actually, D-F-I show that for a reduced ring, an NCCR in this sense is the same thing as an NCCR of the normalization, so you don't get anything new in that case. Gorenstein-ness is a different story.)

It is not clear to me that this is the unique correct answer, but I don't have a better suggestion today.

Part 2. **Sources of NCCRs**

This lecture will describe two classes of examples which serve as sources of non-commutative crepant resolutions. Both of them have already come up in other talks at this workshop, but at the risk of redundancy I will try to discuss them from a slightly different point of view.

1. Quotient singularities of finite groups

Setup.

- $S = k[[x_1, \ldots, x_n]]$, $n \geqslant 2$ (or the [localized] polynomial ring)
- $G \subset \mathrm{GL}_n(k)$ a finite group acting on S by linear changes of variable, with $|G| \in k^\times$.
- $R = S^G$ the ring of invariants.

Example. $S = k[x, y]$, $G = C_m = \langle\sigma\rangle$ where σ multiplies each variable by a primitive m^{th} root of unity (with m not divisible by char k).

Then $R = k[x^m, x^{m-1}y, \ldots, y^m]$. As an R-module, S decomposes as a direct sum

$$\begin{aligned} S &\cong R \oplus R(x, y) \oplus R(x^2, xy, y^2) \oplus \cdots \oplus R(x^{m-1}, \ldots, y^{m-1}) \\ &\cong R \oplus (x^m, x^{m-1}y) \oplus \cdots \oplus (x^m, x^{m-1}y, \ldots, xy^{m-1}) \,. \end{aligned}$$

Observe that R is Gorenstein if and only if $m = 2$.

In general: $G \subset \mathrm{SL}_n(k) \implies G$ is Gorenstein. The converse holds if G is "small", that is, no non-identity element of G fixes a hyperplane.

We always have: that R is a Cohen-Macaulay ring, and S is a MCM R-module.

Aside: MCM modules are the correct representation theory to look at in higher Krull dimension. If you try to put restrictions like "finite representation type" on the set of all modules, you end up with only trivial examples.

Definition. A module M over a local ring $(R, \mathfrak{m}, k)$ is MCM if there is an M-regular sequence of maximal length ($= \dim R$) in $\mathfrak{m}$. Equivalently, $\mathrm{Ext}^i_R(k, M) = 0$ for $i < \dim R$. If R is Gorenstein, equivalent to $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > 0$.

An easier way to think about MCM-ness in this context is that S is fin. gen. free over a polynomial subring of R.

As a natural source of MCM modules, we would like to understand ${}_RS$. Introduce an intermediary:

Definition. The skew group ring (twisted group ring, crossed product ring, ...) $S\#G$ is

- the free S-module on $\{\sigma\}_{\sigma \in G}$
- multiplication defined by $(s\sigma)(t\tau) = s\sigma(t)\sigma\tau$ (twisted!)

Fact: $\operatorname{mod} S\#G \equiv \operatorname{mod}^G S$, the category of G-equivariant finitely generated S-modules.

Define a natural map $\gamma \colon S\#G \longrightarrow \mathrm{End}_R(S)$ by

$$s\sigma \mapsto \qquad \{s\sigma \colon S \longrightarrow S, \quad t \mapsto s\sigma(t)\} .$$

In words, this just considers group elements σ as endomorphisms.

This is a ring homomorphism precisely because of the twisted multiplication in $S\#G$.

Theorem (Auslander '62)**.** *If $G \subset \mathrm{GL}_n(k)$ is small, then γ is an isomorphism.*

Corollary. *If G is small, then R has an algebra $E = \mathrm{End}_R(S)$ which is*

- *of finite global dimension*

- *MCM as an R-module (since it is a free S-module).*

Two special things happen in dimension two.

Theorem (Herzog '78). *If $n = 2$ then every MCM R-module appears as a direct summand of a direct sum of copies of S.*

Proof. For a two-dimensional normal domain, MCM means reflexive. So let M be reflexive. Since $R \hookrightarrow S$ is split ($|G| \in k^\times$) we get

$$M \cong \operatorname{Hom}_R(M^*, R) \overset{\text{split}}{\hookrightarrow} \operatorname{Hom}_R(M^*, S)\,.$$

Since $\operatorname{Hom}_R(M^*, S)$ is an S-module and has depth 2, it's free by Auslander-Buchsbaum. $\square$

Corollary. *If $n = 2$ then R has finite CM representation type.*

This fails horribly if $n \geqslant 3$. However, if R is a CM local ring of finite CM representation type, with indecomposable MCMs $M_1, \ldots, M_r$, then $E = \operatorname{End}_R(\bigoplus M_i)$ has finite global dimension. (It is not necessarily MCM.) [Iyama, Leuschke, Quarles]

Theorem (Kapranov-Vasserot '00). *Assume $G \subset \mathrm{SL}_2(\mathbb{C})$. Let H denote the (unique) minimal resolution of singularities of $\operatorname{Spec} R$. There is an equivalence*

$$D^b(\operatorname{coh} H) \simeq D^b_G(\operatorname{coh} \mathbb{C}^2)\,,$$

where the thing on the right is the bounded derived category of G-equivariant coherent sheaves on $\mathbb{C}^2$. But that's nothing but the bounded derived category of fin. gen. $S\#G$-modules, $D^b(\operatorname{mod} S\#G) \simeq D^b(\operatorname{mod} E)$.

So we can think of $D^b(\operatorname{mod} E)$ (or just E itself) as a resolution of singularities of R!?!

Q: Can one extend Kapranov-Vasserot to $n \geqslant 3$?

Theorem (Bridgeland-King-Reid '01). *For $G \subset \mathrm{SL}_3(\mathbb{C})$, there is a resolution of singularities H such that there are equivalences*

$$D^b(\operatorname{coh} H) \simeq D^b_G(\operatorname{coh} \mathbb{C}^3) \simeq D^b(\operatorname{mod} S\#G)$$

So even though S no longer contains all the MCMs, $E = \operatorname{End}_R(S)$ still "is" the resolution.

Conjecture (BKR, Nakamura, Douglas). $G \subset \mathrm{SL}_n(\mathbb{C})$. *If $Y \longrightarrow \mathbb{C}^n/G = \operatorname{Spec} R$ is a crepant resolution of singularities, $\pi^*\omega_X = \omega_Y$, then*

$$D^b(\operatorname{coh} Y) \simeq D^b_G(\mathbb{C}^n)\,.$$

In particular $D^b(\operatorname{coh} Y) \simeq D^b(\operatorname{mod} \operatorname{End}_R(S))$ is independent of Y!

2. NCCRs from tilting

This example also serves to explain (again) why endomorphism rings are of particular interest.

Definition. A tilting complex $\mathcal{T}$ on a (noetherian, quasi-projective) scheme X is a perfect complex (i.e. a complex isomorphic to a bounded complex of locally free sheaves of finite rank) such that

(1) $\mathcal{T}$ generates all perfect complexes in the sense that the smallest triangulated subcategory of $D^b(\operatorname{coh} X)$ containing $\mathcal{T}$ and closed under finite sums, summands, shifts, and triangles (2 out of 3) contains all perfect complexes.
(2) $\operatorname{Hom}_{D^b(\operatorname{coh} X)}(\mathcal{T}, \mathcal{T}[i]) = 0$ for all $i \neq 0$.

If $\mathcal{T}$ is a single vector bundle (rather than a complex of such) then we call it a tilting bundle.

Theorem (Fundamental Theorem of (Geometric) Tilting). *Let Y be a nice scheme (projective over an affine base) and let $\mathcal{T}$ be a tilting bundle. Set $\Lambda = \operatorname{End}_X(\mathcal{T})$. Then*

(1) The functor $\mathbf{R}\mathrm{Hom}_X(\mathcal{T}, -) \colon D^b(X) \longrightarrow D^b(\Lambda)$ *is an equivalence.*
(2) Y *is smooth if and only if* $\operatorname{gldim} \Lambda < \infty$*.*

This suggests a strategy for constructing NC(C)Rs: Given X, find a resolution of singularities Y, and construct a tilting bundle $\mathcal{T}$ on Y. Then in good situations $\mathrm{End}_Y(\mathcal{T}) = \mathrm{End}_X(p_*\mathcal{T})$ will be an $\mathcal{O}_X$-algebra, and will have finite global dimension. We only have to worry about whether it is MCM.

Theorem (Iyama-Wemyss '11)**.** *Let* R *be a Gorenstein normal domain and set* $X = \operatorname{Spec} R$*. Let* Y *be a normal Gorenstein variety, and let* $\pi \colon Y \longrightarrow X$ *be a projective birational map (for example a resolution of singularities).*

Assume that $\mathcal{T}$ *is a tilting bundle on* Y*, and set* $\Lambda = \mathrm{End}_Y(\mathcal{T})$*. Then the following are equivalent:*

(1) π *is crepant, that is,* $\pi^*\omega_X = \omega_Y$*.*
(2) Λ *is a MCM* R*-module.*

Warning: If R is not Gorenstein, it can still happen that Λ is MCM even though π is no longer crepant. Typically this requires having an explicit handle on a resolution of singularities (which is not very common), and showing that Λ is still MCM is pretty involved. We will see an example below.

This is good news, because in dimensions $\geqslant 4$, resolutions are very rarely crepant.

(One might also take this as a hint that crepancy, and hence MCM-ness, is too strong an assumption.)

2.1. **Determinantal rings.** Here is a direct application of the above, based on joint work with Buchweitz and Van den Bergh.

Setup:

- Let k be a field

- Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates, $m \leqslant n$.
- Set $S = k[X] = x[x_{ij}]$ and $R = S/I_t(X)$, where $I_t(X)$ is the ideal of t-minors of X.

It's known that R is Gorenstein iff $m = n$, in which case R is a hypersurface ring $S/\det(X)$.

Think of $\operatorname{Spec} S = K^{m\times n}$ as the affine space of $m \times n$ matrices over k. Then $\operatorname{Spec} R \subset \operatorname{Spec} S$ is the subset of matrices of rank $< m$.

There exists an explicit, classical, resolution of singularities Z of $\operatorname{Spec} R$. If R is Gorenstein, it is a crepant resolution.

To formalize this, let $G = k^n$, $F = k^m$, and

$$L = \operatorname{Hom}_k(G, F) \cong k^{mn} \cong \operatorname{Spec} S\,.$$

Set

$$Y = \operatorname{Grass}(t-1, F^\vee) \times_k L \cong \operatorname{Grass}(t-1, m) \times_k k^{mn}\,.$$

Then consider the <u>incidence variety</u>

$$\mathcal{Z} = \left\{ (V, \theta) \in \operatorname{Grass}(t-1, F^\vee) \times L \mid \operatorname{image}\theta \subseteq V \right\}\,.$$

Since $\theta \in L$ has rank $< t$ iff its image is contained in a $(t-1)$-plane, the projection $\mathcal{Z} \longrightarrow L \cong \operatorname{Spec} S$ actually lands in $\operatorname{Spec} R$.

<u>Fact:</u> $\mathcal{Z}$ is a resolution of singularities of $\operatorname{Spec} R$, called the "Springer resolution." If R is Gorenstein, it is a crepant resolution.

$$\begin{tikzcd}
\mathcal{Z} \arrow[dr] \arrow[rr, bend left, "p"] \arrow[dd, "q"'] & & \\
& Y = \mathbb{P}(F^\vee) \times L \arrow[r] \arrow[d] & \mathbb{P} = \mathbb{P}(F^\vee) \\
\operatorname{Spec} R \arrow[r, hook] & L = \operatorname{Hom}_k(G, F) &
\end{tikzcd}$$

Theorem (BLV). *There is a (characteristic-free) tilting bundle T on Z such that $\operatorname{End}_Z(T)$ is a MCM R-module and has finite global dimension. If $m = n$, it is a NCCR of R.*

Part 3. Connections with Algebraic Geometry

Let's talk about algebraic geometry. Last time the definition of Van den Bergh had a word in it that came out of nowhere: crepant. Next we want to explain that.

Recall that we wanted to impose a "Gorenstein-ness" assumption on an R-algebra Λ, that $\mathrm{Hom}_R(\Lambda, \omega_R)$ be projective as Λ-module. If R is Gorenstein, this is the same as asking that Λ be symmetric: $\mathrm{Hom}_R(\Lambda, R) \cong \Lambda$. (And we know that this holds if Λ is an endomorphism ring.)

What does this mean in terms of the motivating geometry?

Let X be an algebraic variety, and let ω_X be the canonical sheaf (aka dualizing sheaf, if X is CM) for X.

Suppose we have a resolution of singularities $\pi\colon \widetilde{X} \longrightarrow X$. There is also a canonical sheaf upstairs, $\omega_{\widetilde{X}}$. In fact, by basic properties of the canonical sheaf, we have

$$\omega_{\widetilde{X}} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_{\widetilde{X}}, \omega_X)\,.$$

Now assume that X is Calabi-Yau, that is, $\omega_X \cong \mathcal{O}_X$. This is a strong version of Gorenstein-ness, which would just say that ω_X is locally free. Then the above says

$$\omega_{\widetilde{X}} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_X)\,.$$

and if we assume that $\mathcal{O}_{\widetilde{X}}$ is a symmetric $\mathcal{O}_X$-algebra, then this means that

$$\omega_{\widetilde{X}} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_X) \cong \mathcal{O}_{\widetilde{X}} = \pi^*\mathcal{O}_X = \pi^*\omega_X\,,$$

so that $\widetilde{X}$ is also Calabi-Yau, and the resolution π "preserves the canonical sheaf".

Such resolutions are called crepant. Specifically, π is a crepant resolution if

$$\pi^*\omega_X = \omega_{\widetilde{X}}\,.$$

Informally, this means that the two natural $\mathcal{O}_X$-module structures on $\omega_{\widetilde{X}}$ coincide: the "induced", via $\otimes$, and the "co-induced", via $\mathscr{H}om$.

This is why we defined the non-commutative crepant resolution.

How far does this analogy with algebraic geometry go? Well, here's one thing we would like:

Fact. If a complex ($\mathbb{C}$) algebraic variety X is Gorenstein and has a crepant resolution of singularities, then X has rational singularities.

(This means that $R\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$; equivalently X is normal and $R^i\pi_*\mathcal{O}_{\widetilde{X}} = 0$ for $i > 0$.)

(This follows directly from Grauert-Riemenschneider vanishing, which is why I needed the complex numbers.)

Theorem (Stafford-Van den Bergh)**.** *Let k be an algebraically closed field of characteristic 0 and Δ be a prime affine k-algebra that is finitely generated as a module over its center $Z(\Delta)$. If Δ is a non-singular $Z(\Delta)$-order then $Z(\Delta)$ has only rational singularities.*

In particular, suppose R is a Gorenstein normal affine k-algebra. If R has an NCCR, then $\operatorname{Spec}(R)$ has only rational singularities.

This makes us feel much better.

Historical digression: The results above encourage us to believe that an NCCR is "the same thing" as a crepant resolution of singularities. In fact this was the intuition behind Van den Bergh's original definition, as I will now explain briefly.

The Minimal Model Program is a strategy for carrying out the birational classification of smooth algebraic varieties. The details aren't important for us here. "Running the MMP" on a variety involves certain "moves", which are intended to simplify a variety until it you can't go any farther (you get to a "terminal" variety).

Bondal and Orlov suggest to view the MMP as operations on the bounded derived category of a variety.

Example. One of the moves is blowing up a smooth subvariety. Bondal and Orlov show that this operation induces a fully faithful functor (even a semi-orthogonal decomposition!) on derived categories.

Example. Another of the moves is a "flop". This means replacing a smooth variety Y by Y', where

$$\begin{tikzcd} Y \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

is a diagram of varieties, and f and f' are both crepant resolutions of singularities (and some other technical conditions).

Algebraically this is easy to describe, at least after passing to the completions of the local coordinate rings: a local Gorenstein terminal singularity is a hypersurface ring of multiplicity 2. So the defining equation can be written $u^2 + \cdots$; then Y' is just Y, and f' is the composition of $u \mapsto -u$ with f.

Bondal and Orlov make the following conjecture.

Conjecture. *If Y and Y' are related by a flop, then they are derived equivalent: $D^b(\operatorname{coh} Y) \simeq D^b(\operatorname{coh} Y')$.*

In the case $\dim Y = 3$, this was proved by Bridgeland in 2002, using Fourier-Mukai transforms.

Working through Bridgeland's proof, Van den Bergh observed that Fourier-Mukai transforms are also used in the general approach due to Bridgeland-King-Reid to the McKay correspondence, which includes non-commutative rings as an essential feature. This was his motivation for introducing NCCRs. He proves

Theorem (VdB '04). *Let R be a Gorenstein normal $\mathbb{C}$-algebra, $X = \operatorname{Spec} R$, and $\pi \colon \widetilde{X} \longrightarrow X$ a crepant resolution of singularities. Assume π has at most one-dimensional fibres (e.g. $\dim R \leqslant 3$). Then R has an NCCR $\Lambda = \operatorname{End}_R(M)$.*

Furthermore, Λ is derived equivalent to $\widetilde{X}$.

The Bondal-Orlov conjecture in this case reduces to showing that the corresponding non-commutative rings Λ and, say, Λ', are derived equivalent, which Van den Bergh does, reproving Bridgeland's theorem.

This raises a more general conjecture: Perhaps all crepant resolutions, commutative as well as non-commutative, are derived equivalent. Or (the "non-commutative Bondal-Orlov conjecture") at least all NCCRs of a ring are derived equivalent.

The ncBO conjecture is known for $\dim R \leqslant 3$ [Iyama-Wemyss, first for Gorenstein rings then for CM rings].

Actually, Iyama and Wemyss give a very tempting sufficient condition for the derived-equivalence of two NCCRs $\operatorname{End}_R(M)$ and $\operatorname{End}_R(N)$. Since $\operatorname{End}_R(M)$ has finite global dimension, $\operatorname{Hom}_R(M, N)$ has finite projective dimension. By Yoneda's Lemma, this means that N has a finite resolution by direct sums of direct summands of M, say $0 \longrightarrow M_{\dim R-2} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow N \longrightarrow 0$, which remains exact upon applying $\operatorname{Hom}_R(M, -)$. Say that (M, N) satisfies the depth condition if $\operatorname{Hom}_R(M_i, N)$ locally has depth at least $\dim R - i - 1$ for each i. Then (Iyama-Wemyss prove) $\operatorname{End}_R(M)$ and $\operatorname{End}_R(N)$ to be derived equivalent, it suffices that (M, N) and (N^*, M^*) both satisfy the depth condition.

The condition is clearly vacuous if $\dim R \leqslant 3$.

By the way, it is not known (even in dimension three) whether existence of a crepant resolution of singularities is equivalent to having an NCCR.

It's known to be true under some circumstances in dimension three, and VdB conjectures it to be true generally in dim three, but it's known to be false (both directions) in dimensions $\geqslant 4$.

Example. $S = \mathbb{C}[[x, y, z, t]]$ with C_2 acting by negating the variables. The invariant ring R has a NCCR by the McKay correspondence, but is known not to have a CR [Reid].

Example. Dao gives some examples of hypersurfaces with CRs but no NCCRs, for example, $\mathbb{C}[[x_1, \ldots, x_5]]/(x_1^5 + x_2^4 + x_3^4 + x_4^4 + x_5^4)$. More generally, he shows (using "Tor-rigidity") that a hypersurface ring with an isolated singularity has <u>no</u> NCCR if its dimension is <u>even</u> and at least 4.