WILD HYPERSURFACES

GRAHAM J. LEUSCHKE

ABSTRACT. In the representation theory of finite-dimensional algebras over a field, Drozd's trichotomy theorem says that an algebra has either tame module type or wild module type. Loosely, these two possibilities correspond to: hoping for a classification theorem, or throwing up our hands in dismay. We'd very much like a similar trichotomy result in other representation-theoretic contexts, specifically for maximal Cohen–Macaulay modules over a Cohen–Macaulay local ring. I'll give a little background on the problem, including definitions of tame and wild CM type, and talk about recent work with Andrew Crabbe (Syracuse) which shows that hypersurfaces of multiplicity four or more, in three or more variables, have wild CM type.

The theory of tame and wild representation types is, to some extent, a theorem in search of definitions (and, of course, a proof). Here is a template.

Trichotomy Theorem Template. *Let $\mathcal{C}$ be a module category. Then exactly one of the following holds.*

- *$\mathcal{C}$ has a classification schema like the Fundamental Theorem of Finitely Generated Abelian Groups: the indecomposables are classified by a few discrete parameters (like $(p, n) \rightsquigarrow \mathbb{Z}/p^n\mathbb{Z}$)*
- *$\mathcal{C}$ has a classification schema like the Jordan form: the indecomposables are classified by finitely many discrete parameters (like rank) and one continuous sub-parameter (the eigenvalue).*
- *$\mathcal{C}$ has no classification schema: any classification theorem would involve simultaneously classifying the modules over every other ring as well.*

(Note this is not a theorem, just a template for one.)

We call these options discrete, tame, and wild representation type, respectively. Note that discrete type includes finite type, in which case there are only finitely many indecomposables.

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Key words and phrases. couldn't, drag, me, away.

The first (to my knowledge) precise definitions of tameness and wildness are due to Freislich and Donovan in 1973. They've been modified by many people since then, but consensus seems to have settled on something like the following.

Definition. Let $\mathcal{C}$ be a category of (f.g.) modules over a k-algebra Λ, where k is an infinite field.

- A <u>one-parameter family</u> in $\mathcal{C}$ is a set of modules of the form

  $$\overline{\mathcal{M}} = \{\mathcal{M}/(T-c)\mathcal{M} \;\mid\; c \in k\},$$

  where $\mathcal{M}$ is a Λ-$k[T]$-bimodule which is finitely generated and free over $k[T]$.
- We say $\mathcal{C}$ has <u>tame type</u> if $\mathcal{C} = \bigcup_n \mathcal{C}_n$, and for every n, the indecomposable modules in $\mathcal{C}_n$ form a one-parameter family with maybe finitely many exceptions. (Think of n as k-dimension or rank, possibly subdivided.)
- We say $\mathcal{C}$ has <u>wild type</u> if $\mathcal{C}$ contains n-parameter families of indecomposables for arbitrarily large n.
- Equivalently [this is not obvious], $\mathcal{C}$ has wild type if for every finite-dimensional k-algebra Γ, there is a <u>representation embedding</u> of Γ-mod into $\mathcal{C}$. This means an exact functor from Γ-mod to $\mathcal{C}$ which preserves non-isomorphy and indecomposability. [Sometimes people also insist that the functor be fully faithful, i.e. a bijection on Hom-sets.]

(Technically, note that tame type as defined includes finite type; we still talk about trichotomy rather than dichotomy.)

Theorem (Drozd 1977, Crawley-Boevey 1988)**.** *Let Λ be a finite-dimensional algebra over an algebraically closed field. Then Λ-mod has either tame type or wild type, and not both.*

Example. Here is the standard first example. Let $\Lambda = k\langle a, b\rangle$, the non-commutative polynomial ring in two variables. Then the category of Λ-modules which are finite-dimensional over k has wild type.

Specifically, let $\Gamma = k\langle x_1, \ldots, x_m\rangle$ be any finite-dimensional k-algebra (the x_i need not be algebraically independent – there may be some relations), and let V be any Γ-module of k-dimension n. Then the action of Γ on V can be represented by $n \times n$ matrices $X_1, \ldots, X_m$ over k. Let $c_1, \ldots, c_m$ be distinct scalars in k, and define $mn \times mn$ matrices

$$A = \begin{bmatrix} X_1 & & & \\ \mathrm{id}_V & X_2 & & \\ & \ddots & \ddots & \\ & & \mathrm{id}_V & X_m \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} c_1 \,\mathrm{id}_V & & & \\ & c_2 \,\mathrm{id}_V & & \\ & & \ddots & \\ & & & c_m \,\mathrm{id}_V \end{bmatrix}.$$

Use these to define a Λ-action on $M = V^{(m)}$, so a acts as A and b acts as B. One shows (by relatively simple matrix analysis) that this functor $V \rightsquigarrow M$ embeds the finite-dimensional Γ-modules into $k\langle a, b\rangle$-mod, preserving non-isomorphism and indecomposability. It's even surjective on Hom-sets.

In slightly more detail, any Λ-homomorphism between $M = V^{(m)}$ and $N = W^{(m)}$ is given by a k-homomorphism $\Phi = (\varphi_{ij})_{i,j \leqslant n} \colon V^{(m)} \longrightarrow W^{(m)}$ such that $A_W \Phi = \Phi A_V$ and $B_W \Phi = \Phi B_V$. The second equation is always true, so imposes no restrictions. One shows that the first equation implies that Φ is a diagonal matrix with $\varphi_{11} = \cdots = \varphi_{mm}$. Since the functor $k - mod \longrightarrow \Lambda - mod$ sending V to $V^{(m)}$ sends linear maps to diagonal matrices, this shows that it is full and faithful.

Note that in fact it would have been enough — if all we wanted was to preserve non-isomorphy and indecomposability — to show that Φ is lower-triangular with $\varphi_{11} = \cdots = \varphi_{mm}$, for then Φ would be an isomorphism/split surjection if and only if φ_{11} was so.

Example (Gel'fand-Ponomarev '69, Drozd '72, Levy-Klingler '05)**.** The modules of finite length over either the commutative polynomial ring $k[x, y]$ or the power series ring $k[[x, y]]$ have wild type.

More specifically, let $R = k[x, y]/(x^2, xy^2, y^3)$, a local ring of length 5. [Klingler and Levy call this the "Drozd ring".] Let $\Gamma = k\langle a, b\rangle$ be and V an n-dimensional Γ-module, with $n \times n$ matrices A and B defining the action of Γ. Then there are $32n \times 32n$ matrices X and Y defining an action of R on $M = V^{(32)}$, and this functor shows that R-mod has wild type. Therefore the power series and polynomial rings do too, since finite-dimensional modules over R are also finite-dimensional over those rings.

In case you ever want to know those $32n \times 32n$ matrices, here they are. We have our $n \times n$ matrices A and B; define a $5n \times 5n$ matrix C and a $2n \times 5n$ matrix D by

$$C = \begin{bmatrix} c_1 \operatorname{id}_V & & & \\ & c_2 \operatorname{id}_V & & \\ & & \ddots & \\ & & & c_5 \operatorname{id}_V \end{bmatrix}, \qquad D = \begin{bmatrix} \operatorname{id}_V & 0 & \operatorname{id}_V & \operatorname{id}_V & \operatorname{id}_V \\ 0 & \operatorname{id}_V & \operatorname{id}_V & A & B \end{bmatrix}.$$

Then we set

$$X = \begin{bmatrix} 0 & 0 & I_{15} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} Y_1 & 0 & Y_2 \\ 0 & 0 & Y_3 \\ 0 & 0 & Y_1 \end{bmatrix},$$

where

$$Y_1 = \begin{bmatrix} 0 & 0 & I_5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad Y_2 = \begin{bmatrix} 0 & 0 & 0 \\ I_5 & 0 & 0 \\ 0 & B & 0 \end{bmatrix}, \qquad Y_3 = \begin{bmatrix} 0 & C & 0 \end{bmatrix}$$

Note the blocks of X and Y are not uniform sizes: the top row of each is $15n$, the second is $2n$, and the third is $15n$ again. The columns follow the same pattern.

One verifies that $X^2 = XY^2 = Y^3 = 0$, so that M really is an R-module. Then one shows that a homomorphism φ between two such modules is an isomorphism if and only if φ is a scalar matrix: diagonal with the same constant down the diagonal.

It follows that the finite-length modules over $k[a_1, \ldots, a_n]$ and $k[[a_1, \ldots, a_n]]$ have wild type as well. We'll use this below.

Here's one example of tame type, just so you know they exist.

Example (Kronecker 1896)**.** Let $R = k[a,b]/(a^2, b^2)$. Then classifying finite-dimensional R-modules is equivalent to the problem of classifying pairs of square matrices (A, B) up to simultaneous equivalence: $(A, B) \sim (PAQ, PBQ)$. Kronecker worked this out: there are two indecomposables of each odd k-dimension, and a $\mathbb{P}^1$ worth of modules in each even dimension.

Theorem (Klingler-Levy)**.** *A complete local (commutative) ring is finite-length wild if and only if either (i) it maps onto the Drozd ring; (ii) it maps onto the path algebra of the quiver with three parallel arrows. (These are called "minimal wild rings".)*

Now let's talk about maximal Cohen–Macaulay modules over a Cohen-Macaulay local ring. These are the modules M such that $\operatorname{depth} M = \dim R$. If $\dim R = 1$ and R is reduced, they are just the torsion-free modules; if $\dim R = 2$ and R is normal, they are the reflexive modules.

We adopt the definitions of discrete, finite, tame, and wild type verbatim for this context, and call them discrete CM type, finite CM type, tame CM type and wild CM type. (Clarification: for wild type, we still ask for a representation embedding of the finite-length Γ-modules, not MCM modules over some ring.)

The situation is most satisfactory in dimension one. Assume $\operatorname{char} k = 0$ and $k = \overline{k}$ for simplicity.

Theorem (Drozd-Greuel '92)**.** *The Trichotomy Theorem holds for complete CM local rings of dimension one containing k.*

The corresponding statement for rings of dimension $d \geqslant 2$ is open. I have been told that Drozd has recently proved a trichotomy theorem for two-dimensional domains: quotient singularities (rational singularities) have finite CM type; log-canonical singularities have tame CM type; and all others have wild CM type. I haven't seen the details.

The proof of this theorem is in terms of matrix problems, or BOCSes. In particular, it gives no immediate information about any particular ring. The key to understanding particular rings lies in the so-called "ADE" or "simple" hypersurfaces.

Definition. The <u>one-dimensional ADE hypersurfaces</u> are the hypersurface rings $k[\![x,y]\!]/(f(x,y))$ defined by the following polynomials.

(A_n) $x^2 + y^{n+1}$, $n \geqslant 1$
(D_n) $x^2y + y^{n-1}$, $n \geqslant 4$
(E_6) $x^3 + y^4$
(E_7) $x^3 + xy^3$
(E_8) $x^3 + y^5$.

Theorem (Greuel-Knorrer 1985)**.** *A reduced complete one-dimensional local $\mathbb{C}$-algebra has finite CM type if and only if it <u>birationally dominates</u> one of the one-dimensional ADE hypersurfaces. (Recall that S birationally dominates R if $R \subseteq S \subseteq \overline{R}$.) A complete one-dimensional hypersurface over $\mathbb{C}$ has finite CM type if and only if it is isomorphic to one of the ADE hypersurfaces.*

For example, $k[[t^3, t^4, t^5]]$ birationally dominates $(E_6) \cong k[[t^3, t^4]]$ and $(E_8) \cong k[[t^3, t^5]]$, so has finite CM type. On the other hand, $k[[t^5, t^7, t^8]]$ has multiplicity 5, so does not birationally dominate any of the ADE hypersurfaces, so has infinite CM type.

The upshot here is that for finite CM type in dimension one, the ADE hypersurfaces call the tune. In fact, they control all of discrete type: their "natural limits," the (A_∞) hypersurface x^2 and the (D_∞) hypersurface x^2y are the only hypersurfaces of discrete but infinite CM type.

What about tame type? Is there a family of hypersurfaces that control the situation similarly? Yes.

Definition. The <u>one-dimensional T_{pq} hypersurfaces</u> are the rings $k[\![x,y]\!]/(T_{pq}(x,y))$, where

$$T_{pq}(x,y) = x^p + \lambda x^2y^2 + y^q\,,$$

with $p,q \geqslant 2$ and $\lambda \in \mathbb{C} \setminus \{0,1\}$. (Actually, for $(p,q) \neq (3,6)$ or $(4,4)$, the parameter λ can be omitted, since all values of λ give isomorphic rings, even $\lambda = 1$. For those special pairs (p,q), the λ is needed. Notice that those are the pairs such that $1/p + 1/q = 1/2$.)

Theorem (Drozd-Greuel '93)**.** *The one-dimensional T_{pq} hypersurfaces have tame CM type. Furthermore, a reduced complete one-dimensional local $\mathbb{C}$-algebra has tame CM type if and only if it birationally dominates one of the T_{pq} hypersurfaces.*

The proof is interesting: they actually prove that $P_{pq} = k[x, y, z]/(xy, x^p + y^q + z^2)$ has tame type, using some deformation theory. But this is the endomorphism ring of the maximal ideal of the corresponding T_{pq}, so T_{pq} has tame type too.

These results give a very clear picture of the Trichotomy Theorem for MCM modules in dimension one.

So we've seen that hypersurfaces call the tune in dimension one; what else can we say about hypersurfaces? As far as discrete type goes, once again the ADEs are in control.

Theorem (Knorrer '87)**.** *A (non-regular) complete hypersurface ring over a field $R = k[[x_1, \dots, x_n]]/(f)$ (k alg. closed, char. 0) has finite CM type if and only if $R \cong k[[x, y, z_3 \dots, x_n]]/(g(x, y) + z_3^2 + \cdots + z_n^2)$, where g is one of the ADE hypersurfaces. It has countable CM type if and only if it has that form with g either (A_∞) or (D_∞).*

One key step in the proof of this theorem is that if a hypersurface $k[[x_1, \dots, x_n]]/(f)$ with $n \geqslant 3$ has multiplicity 3 or more (that is, $f \in (x_1, \dots, x_n)^3$), then we can construct a $\mathbb{P}^1$ worth of nonisomorphic indecomposable MCM modules. Thus, if the hypersurface has finite (or even discrete) CM type, then it has multiplicity at most 2. Therefore it has a linear or quadratic term, and you can complete the square to write it in the form $g + z_n^2$, where g does not involve the variable z_n.

What about tame CM type?

Theorem (Drozd-Greuel-Kashuba, 2002; Burban-Drozd, 2010)**.** *The two-dimensional hypersurface rings defined by*

$$T_{pqr}(x, y, z) = x^p + y^q + z^r + \lambda xyz \, ,$$

where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leqslant 1$ and $\lambda \in \mathbb{C}^*$, have tame CM type. (In the case $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, we may assume $\lambda = 1$.) Burban and Drozd recently proved that we can take *any of p,q,r to be ∞, so that $T_{\infty\infty\infty} = xyz$ has tame type, as does $T_{23\infty} = x^2 + y^3 + xyz$.*

To obtain a complete classification of the hypersurfaces of tame type, we would need a result like the "key step" above: high multiplicity implies wildness. The "key step" itself doesn't work, since it only constructs a single one-parameter family of indecomposable modules. There is one previous result in this direction:

Theorem (Bondarenko 2007)**.** *Let $f \in k[[x,y,z]]$ have order $e \geqslant 4$. Then $k[[x,y,z]]/(f)$ has wild CM type.*

Our modest contribution is to extend this last fact.

Theorem (Crabbe-Leuschke 2010)**.** *Let $f \in k[[x_1,\dots,x_n,z]]$, $n \geqslant 2$, have order $e \geqslant 4$. Then $R = k[[x_1,\dots,x_n,z]]/(f)$ has wild CM type.*

Before sketching the proof, recall that for hypersurface rings, MCM modules are completely given by <u>matrix factorizations</u>: pairs of square matrices (φ,ψ) such that $\varphi\psi = \psi\varphi = f \cdot \mathrm{id}$. MCM modules over hypersurfaces have periodic free resolutions of period 2 [Eisenbud 1980], and φ and ψ are the matrices in the resolution.

Example. Consider

$$f = x_1y_1 + x_2y_2 + \cdots + x_ny_n \in k[[x_1,\dots,x_n,y_1,\dots,y_n]] = S\,.$$

Then $S/(f)$ is isomorphic to an (A_1) singularity of dimension $2n-1$, and has exactly one non-free indecomposable MCM module, with matrix factorization given by

$$(\varphi_1,\psi_1) = ([x_1]\,,[y_1])$$

if $n=1$, and inductively

$$(\varphi_n,\psi_n) = \left(\begin{bmatrix} \varphi_{n-1} & -y_n\,\mathrm{id}_{2^{n-1}} \\ x_n\,\mathrm{id}_{2^{n-1}} & \psi_{n-1}\end{bmatrix}, \begin{bmatrix} \psi_{n-1} & y_n\,\mathrm{id}_{2^{n-1}} \\ -x_n\,\mathrm{id}_{2^{n-1}} & \varphi_{n-1}\end{bmatrix}\right)\,.$$

Sketch of Proof. [1] Introduce parameters $a_1, \dots, a_n$. Then we can write

$$f = z^2 h + (x_1 - a_1 z)g_1 + \cdots + (x_n - a_n z)g_n$$

where h has order at least 2 and each g_i has order at least 3. (This boils down to the fact that $(z^2) + I\mathfrak{m} = \mathfrak{m}^2$, where I is generated by all the $(x_i - a_i z)$.)

This has the shape of an (A_1)-singularity, so we know at least one matrix factorization of f! The entries in the matrices involve the parameters $a_1, \dots, a_n$, so maybe write them as

$$(\varphi_n(\underline{a}), \psi_n(\underline{a}))\,.$$

If, for example, we're in the case $n = 1$, we have

$$(\varphi_1(a_1), \psi_1(a_1)) = \left(\begin{bmatrix} z^2 & -g_1 \\ x_1 - a_1 z & h \end{bmatrix}, \begin{bmatrix} h & g_1 \\ -x_1 + a_1 z & z^2 \end{bmatrix} \right)$$

and if $n = 2$, we have

$$(\varphi_2(a_1, a_2), \psi_2(a_1, a_2)) =$$
$$\left(\begin{bmatrix} z^2 & -g_1 & -g_2 & \\ x_1 - a_1 z & h & & -g_2 \\ x_2 - a_2 z & & h & g_1 \\ & x_2 - a_2 z & -x_1 + a_1 z & z^2 \end{bmatrix}, \begin{bmatrix} h & g_1 & g_2 & \\ -x_1 + a_1 z & z^2 & & g_2 \\ -x_2 + a_2 z & & z^2 & -g_1 \\ & -x_2 + a_2 z & x_1 - a_1 z & h \end{bmatrix} \right).$$

We'll define a representation embedding of the finite-k-dimensional $k[a_1, \dots, a_n]$-modules into the MCM R-modules. Let V be such a module, of k-dimension m. Then the action is given by square commuting matrices $A_1, \dots, A_m$ over k. In $(\varphi_n(\underline{a}), \psi_n(\underline{a}))$, "inflate" each a_i to the matrix A_i. (This amounts to tensoring the matrix factorization with V.)

The result, $(\varphi_n(\underline{A}), \psi_n(\underline{A}))$, is still a matrix factorization (since the A_i commute), now of size $m \cdot 2^{n-1}$. Therefore it defines a MCM module

$$M(\underline{A}) = \operatorname{cok}(\varphi_n(\underline{A}), \psi_n(\underline{A}))\,.$$

Pleasantly dull matrix calculations reveal that $M(\underline{A})$ is indecomposable if V is, and preserves non-isomorphism, so we get a representation embedding. $\square$

[1] expand me