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Some extensions of theorems of Knörrer and Herzog–Popescu

Alex S. Dugas [a], Graham J. Leuschke [b,*]

[a] Dept. of Mathematics, University of the Pacific, Stockton, CA 95211, USA
[b] Dept. of Mathematics, Syracuse University, Syracuse NY 13244, USA

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ABSTRACT

A construction due to Knörrer shows that if N is a maximal Cohen–Macaulay module over a hypersurface defined by $f + y^2$, then the first syzygy of N/yN decomposes as the direct sum of N and its own first syzygy. This was extended by Herzog–Popescu to hypersurfaces $f + y^n$, replacing N/yN by $N/y^{n-1}N$. We show, in the same setting as Herzog–Popescu, that the first syzygy of N/y^kN is always an extension of N by its first syzygy, and moreover that this extension has useful approximation properties. We give two applications. First, we construct a ring Λ over which every finitely generated module has an eventually 2-periodic projective resolution, prompting us to call it a "non-commutative hypersurface ring". Second, we give upper bounds on the dimension of the stable module category (a.k.a. the singularity category) of a hypersurface defined by a polynomial of the form $x_1^{a_1} + \cdots + x_d^{a_d}$.

* Corresponding author.
E-mail addresses: adugas@pacific.edu (A.S. Dugas), gjleusch@syr.edu (G.J. Leuschke).
URL: http://www.leuschke.org/ (G.J. Leuschke).

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1. Introduction

Let S be a regular local ring of dimension $d+1$, and let f be a non-zero non-unit of S. Let $R = S/(f)$ be the d-dimensional hypersurface ring. Fix an integer $n \geqslant 2$ and set $R^{\#} = S[y]/(f + y^n)$. We refer to $R^{\#}$ as the *(n-fold) branched cover of S branched along R* or, by abuse of language, as the n-fold branched cover of R.

We consider maximal Cohen–Macaulay (MCM) modules over R and $R^{\#}$. Our starting point is the following theorem of Knörrer and its generalization by Herzog–Popescu. Write Ω for the syzygy operator.

Theorem 1.1 *([10]). Suppose that $S = k[\![x_0, \dots, x_d]\!]$ and the characteristic of k is not equal to 2. Assume $n = 2$, so that $R^{\#} = S[y]/(f+y^2)$. Then for any MCM $R^{\#}$-module N, we have*

$$\Omega_{R^{\#}}(N/yN) \cong N \oplus \Omega_{R^{\#}}(N)\,.$$

Theorem 1.2 *([7, Theorem 2.6]). Suppose that $S = k[\![x_0, \dots, x_d]\!]$ and the characteristic of k does not divide n. Then for any MCM $R^{\#}$-module N, we have*

$$\Omega_{R^{\#}}(N/y^{n-1}N) \cong N \oplus \Omega_{R^{\#}}(N)\,.$$

We remark that each of these results is a special case of the more general results in the respective papers, the proofs of which do require the additional assumption that S be a power series ring over a field. As a byproduct of our results below, we will see that both results continue to hold true when S is an arbitrary regular local ring.

In this note we consider the full subcategories of $R^{\#}$-modules of the form $\Omega_{R^{\#}}(M)$, where M is annihilated by y^k, for $k = 1, \dots, n$. Specifically, set $A_k = S[y]/(f, y^k)$ for $1 \leqslant k \leqslant n$, and define

$$\Sigma_k = \operatorname{add}\{\Omega_{R^{\#}}(M) \,|\, M \text{ is a MCM } A_k\text{-module}\}\,.$$

We are interested in the categories $\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots \subseteq \operatorname{MCM}(R^{\#})$. Observe that the results of Knörrer and Herzog–Popescu above imply $\Sigma_1 = \operatorname{MCM}(R^{\#})$ when $n = 2$, and $\Sigma_{n-1} = \operatorname{MCM}(R^{\#})$ for arbitrary n, respectively.

Our main result is the following. Recall that a subcategory $\mathcal{X} \subseteq \mathcal{A}$ is called *functorially finite* if every object of $\mathcal{A}$ admits both left and right approximations by objects in $\mathcal{X}$. See Section 2 for precise definitions.

Theorem A. *The subcategories $\Sigma_1, \dots, \Sigma_n$ are functorially finite in* $\operatorname{MCM}(R^{\#})$*. Namely, for every MCM $R^{\#}$-module N there are short exact sequences*

$$0 \longrightarrow \Omega_{R^{\#}}(N) \longrightarrow \Omega_{R^{\#}}(N/y^kN) \longrightarrow N \longrightarrow 0$$

and

$$0 \longrightarrow N \longrightarrow \Omega_{R\#}(\Omega_{R\#}(N)/y^k\Omega_{R\#}(N)) \longrightarrow \Omega_{R\#}(N) \longrightarrow 0$$

for $k = 1, \ldots, n$, which are right and left Σ_k-approximations of N, respectively.

We give two applications of Theorem A.

First we extend a result of Iyama [9, 5.4.1], Leuschke [11, Prop. 8], and Quarles [16, Theorem 3.5.3] to the effect that the endomorphism ring of a direct sum of a complete set of representatives for the isomorphism classes of the MCM modules over a simple (ADE) hypersurface singularity has finite global dimension. See Section 3 for a list of the defining equations of the ADE hypersurface rings. When $f \in k[\![x_0, \ldots, x_d]\!]$ defines a simple singularity of dimension $d \geqslant 1$, the polynomial $f + y^n$ does not in general define a simple singularity for $n \geqslant 3$, so that in particular the branched cover does not have a representation generator. However we show that the n-fold branched cover admits a MCM module M such that $\Lambda = \mathrm{End}_{R\#}(M)$ behaves like a "non-commutative hypersurface ring," in the sense that Λ is Iwanaga–Gorenstein and every finitely generated Λ-module has a projective resolution which is eventually periodic of period at most 2.

Second we give upper bounds on the dimension, in the sense of Rouquier, of the stable category of MCM modules over a complete hypersurface ring defined by a polynomial of the form $x_0^{a_0} + \cdots + x_d^{a_d}$. These bounds sharpen the general bounds of Ballard–Favero–Katzarkov [2, Proposition 4.11] for complete isolated hypersurface singularities in this particular case.

Notation. Unless otherwise specified, all modules are left modules.

For a ring Γ and a left Γ-module X, we write $\Omega_\Gamma(X)$ for an arbitrary first syzygy of X. This is uniquely defined up to projective summands. If Γ is commutative local, $\Omega_\Gamma(X)$ is unique up to isomorphism.

2. Approximating MCM modules over the branched cover

We make essential use of the following general lemma.

Lemma 2.1. *Let Γ be a ring and X a Γ-module. Let $z \in \Gamma$ be a non-zerodivisor on X. Then there is a short exact sequence*

$$0 \longrightarrow \Omega_\Gamma(X) \longrightarrow \Omega_\Gamma(X/zX) \longrightarrow X \longrightarrow 0 \tag{2.1}$$

of Γ-modules.

Proof. Let $0 \longrightarrow \Omega_\Gamma(X) \longrightarrow F \longrightarrow X \longrightarrow 0$ be a short exact sequence of Γ-modules with F free. Multiplication by z on X induces a pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_\Gamma(X) \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "z"] & 0 \\
0 \arrow[r] & \Omega_\Gamma(X) \arrow[r] & F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & & X/zX \arrow[r, equal] \arrow[d] & X/zX \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

the middle column of which shows that $P \cong \Omega_\Gamma(X/zX)$. □

Lemma 2.2. *Let Q be a commutative Noetherian ring, X a finitely generated Q-module, and z a non-zerodivisor on X. The following conditions are equivalent.*

(i) $\Omega_Q(X/zX) \cong X \oplus \Omega_Q(X)$;
(ii) *z annihilates* $\operatorname{Ext}^1_Q(X, \Omega_Q(X))$;
(iii) *z annihilates* $\operatorname{Ext}^1_Q(X, Y)$ *for every finitely generated Q-module Y.*

Proof. Indeed, the short exact sequence (2.1) is the image in $\operatorname{Ext}^1_Q(X, \Omega_Q(X))$ of $z \cdot \mathrm{id}_X \in \operatorname{End}_Q(X)$. Therefore z kills $\operatorname{Ext}^1_Q(X, \Omega_Q(X))$ if and only if (2.1) splits, which by Miyata's theorem [13] happens if and only if $\Omega_Q(X/zX) \cong X \oplus \Omega_Q(X)$. The equivalence of the second and third conditions is well known. □

Now we consider a hypersurface ring R and the n-fold branched cover $R^\#$. As in the introduction, let S be a regular local ring of dimension $d+1$, let f be a non-zero non-unit of S, and let $R = S/(f)$ be the d-dimensional hypersurface ring. Fix $n \geqslant 2$ and set $R^\# = S[y]/(f + y^n)$. Then Lemma 2.1 yields the following.

Proposition 2.3. *Let N be a MCM $R^\#$-module. For each $k \geqslant 1$, we have:*

(i) *There are short exact sequences*

$$0 \longrightarrow \Omega_{R^\#}(N) \longrightarrow \Omega_{R^\#}(N/y^kN) \longrightarrow N \longrightarrow 0 \tag{2.2}$$

and

$$0 \longrightarrow N \longrightarrow \Omega_{R^\#}(\Omega_{R^\#}(N)/y^k\Omega_{R^\#}(N)) \longrightarrow \Omega_{R^\#}(N) \longrightarrow 0\,. \tag{2.3}$$

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(ii) *We have* $\Omega_{R^\#}(N/y^kN) \cong N \oplus \Omega_{R^\#}(N)$ *if and only if* y^k *annihilates* $\operatorname{Ext}^1_{R^\#}(N, \Omega_{R^\#}(N))$.

Proof. Existence of (2.2) and the second statement both follow immediately from Lemma 2.1. For (2.3) it suffices to observe that since $R^\#$ is a hypersurface, if N has no free direct summands, N is isomorphic to its own second syzygy. If on the other hand N is free, then both exact sequences degenerate to the isomorphism $R^\# \cong \Omega_{R^\#}(R^\#/(y^k))$. $\square$

We emphasize that $f \neq 0$ is required for this result, since we need y to be a non-zerodivisor on $R^\#$.

Remark 2.4. We observe that the results of Knörrer and Herzog–Popescu from the introduction hold true in our level of generality. The proof of [7, Theorem 2.6] uses the *Noether different* $\mathcal{N}_A^B$ of a ring homomorphism $A \longrightarrow B$. Let $\mu \colon B \otimes_A B \longrightarrow B$ denote the multiplication map, and define

$$\mathcal{N}_A^B = \mu\left(\operatorname{Ann}_{B\otimes_A B}(\ker \mu)\right) .$$

The two results we need about the Noether different are contained in [3] in the generality required.

First, [3, Theorem 7.8.3] implies that if $A \longrightarrow B$ is a homomorphism of commutative Noetherian rings of finite Krull dimension, with A regular and B Gorenstein, and such that B is a finitely generated free A-module, then $\mathcal{N}_A^B$ annihilates all Ext-modules $\operatorname{Ext}^1_B(X, Y)$ with X, Y finitely generated B-modules.

The second result is contained in [3, Example 7.8.5]. If A is an arbitrary commutative ring and $B = A[y]/(g(y))$ for some polynomial $g(y) \in A[y]$, then the (formal) derivative $g'(y)$ is contained in $\mathcal{N}_A^B$.

Proposition 2.5. *Let S be an arbitrary regular local ring, $f \in S$ a non-zero non-unit, $R = S/(f)$, and $R^\# = S[y]/(f + y^n)$ for some $n \geqslant 1$. Assume that n is a unit in S. Then for any MCM $R^\#$-module N, we have*

$$\Omega_{R^\#}(N/y^{n-1}N) \cong N \oplus \Omega_{R^\#}(N) .$$

Proof. By Proposition 2.3(ii), it is enough to show that y^{n-1} annihilates $\operatorname{Ext}^1_{R^\#}(N, \Omega_{R^\#}(N))$. Since $\frac{\partial}{\partial y}(f+y^n) = ny^{n-1}$ and n is invertible, this follows from Remark 2.4. $\square$

Next we want to consider the relationship between the short exact sequence (2.2) and the module N. We recall here some definitions from “relative homological algebra”.

Definition 2.6. Let $\mathcal{A}$ be an abelian category, $\mathcal{X} \subseteq \mathcal{A}$ a full subcategory, and M an object of $\mathcal{A}$.

(i) A *right $\mathcal{X}$-approximation* (or *$\mathcal{X}$-precover*) of M is a morphism $q\colon X \longrightarrow M$, with $X \in \mathcal{X}$, such that any $f\colon X' \longrightarrow M$ with $X' \in \mathcal{X}$ factors through q. Equivalently, the induced homomorphism $\operatorname{Hom}_{\mathcal{A}}(X', X) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(X', M)$ is surjective.

(ii) A *left $\mathcal{X}$-approximation* (or *$\mathcal{X}$-preenvelope*) is a morphism $j\colon M \longrightarrow X$ with $X \in \mathcal{X}$ such that any $g\colon M \longrightarrow X'$ with $X' \in \mathcal{X}$ factors through j. Equivalently the induced homomorphism $\operatorname{Hom}_{\mathcal{A}}(X, X') \longrightarrow \operatorname{Hom}_{\mathcal{A}}(M, X')$ is surjective.

(iii) A right $\mathcal{X}$-approximation $q\colon X \longrightarrow M$ is *minimal* (or a *$\mathcal{X}$-cover*) if every endomorphism $\varphi \in \operatorname{End}_{\mathcal{A}}(X)$ with $\varphi q = q$ is an isomorphism. Dually, a left $\mathcal{X}$-approximation $j\colon M \longrightarrow X$ is *minimal* (a *$\mathcal{X}$-envelope*) if every endomorphism $\varphi \in \operatorname{End}_{\mathcal{A}}(X)$ with $j\varphi = j$ is an isomorphism.

(iv) We say that $\mathcal{X}$ is *contravariantly finite* (resp. *covariantly finite*) in $\mathcal{A}$ if every object M of $\mathcal{A}$ has a right (resp. left) $\mathcal{X}$-approximation.

(v) A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is *$\mathcal{X}$-exact* (resp. *$\mathcal{X}$-coexact*) if the induced sequence $\operatorname{Hom}_{\mathcal{A}}(X, A) \xrightarrow{f_*} \operatorname{Hom}_{\mathcal{A}}(X, B) \xrightarrow{g_*} \operatorname{Hom}_{\mathcal{A}}(X, C)$ (resp. $\operatorname{Hom}_{\mathcal{A}}(C, X) \xrightarrow{g^*} \operatorname{Hom}_{\mathcal{A}}(B, X) \xrightarrow{f^*} \operatorname{Hom}_{\mathcal{A}}(A, X)$) is exact for all $X \in \mathcal{X}$.

Approximations need not exist, and even when they do, minimal approximations need not exist.

We routinely abuse language by referring to the object X as an approximation of M (on the appropriate side). Furthermore, when $\mathcal{A}$ is a module category and a right (resp. left) $\mathcal{X}$-approximation happens to be surjective (resp. injective) we will refer to the whole short exact sequence $0 \longrightarrow \ker q \longrightarrow X \xrightarrow{q} M \longrightarrow 0$ (resp. $0 \longrightarrow M \xrightarrow{j} X \longrightarrow \operatorname{cok} j \longrightarrow 0$) as the approximation.

The next result contains Theorem A from the Introduction. Recall the definition of Σ_k:

$$\Sigma_k = \operatorname{add}\left\{\Omega_{R^\#}(M) \,|\, M \text{ is a MCM } A_k\text{-module}\right\},$$

where $A_k = S[y]/(f, y^k)$.

Theorem 2.7. *Let N be a MCM $R^\#$-module. The short exact sequence (2.2) is a right Σ_k-approximation of N, and (2.3) is a left Σ_k-approximation of N.*

Proof. To show that (2.2) is a Σ_k-approximation, it suffices to show that an arbitrary $R^\#$-homomorphism $\sigma\colon M \longrightarrow N$, for $M \in \Sigma_k$, factors through $\Omega_{R^\#}(N/y^kN)$. Equivalently, σ maps to zero in $\operatorname{Ext}^1_{R^\#}(M, \Omega_{R^\#}(N))$ in the long exact sequence

$$\operatorname{Hom}_{R^\#}(M, \Omega_{R^\#}(N/y^kN)) \longrightarrow \operatorname{Hom}_{R^\#}(M, N) \longrightarrow \operatorname{Ext}^1_{R^\#}(M, \Omega_{R^\#}(N))$$

obtained by applying $\operatorname{Hom}_{R^\#}(M, -)$ to (2.2).

Recall that (2.2) is the image of the short exact sequence $\chi = (0 \longrightarrow \Omega_{R^\#}(N) \longrightarrow F \longrightarrow N \longrightarrow 0) \in \operatorname{Ext}^1_{R^\#}(N, \Omega_{R^\#}(N))$ under multiplication by y^k. It therefore suffices to show that $y^k\chi$ maps to zero in $\operatorname{Ext}^1_{R^\#}(M, \Omega_{R^\#}(N))$. Since $M \in \Sigma_k$, we can write $M \oplus M' \cong \Omega_{R^\#}(X)$ for an $R^\#$-module X annihilated by y^k and some $R^\#$-module M'. Then

$$\begin{aligned}\operatorname{Ext}^1_{R^\#}(M \oplus M', \Omega_{R^\#}(N)) &= \operatorname{Ext}^1_{R^\#}(\Omega_{R^\#}(X), \Omega_{R^\#}(N)) \\ &\cong \operatorname{Ext}^2_{R^\#}(X, \Omega_{R^\#}(N))\end{aligned}$$

is annihilated by y^k since X is, which proves the claim.

The statement about left approximation follows similarly, using the fact that

$$\begin{aligned}\operatorname{Ext}^1_{R^\#}(N, M \oplus M') &= \operatorname{Ext}^1_{R^\#}(N, \Omega_{R^\#}(X)) \\ &\cong \operatorname{Ext}^2_{R^\#}(\Omega_{R^\#}(N), \Omega_{R^\#}(X)) \\ &\cong \operatorname{Ext}^1_{R^\#}(\Omega_{R^\#}(N), X)\end{aligned}$$

is also killed by y^k. $\square$

Standard arguments about approximations now yield the following.

Corollary 2.8. *For a MCM $R^\#$-module N and $k \geqslant 1$, the following are equivalent:*

(1) $N \in \Sigma_k$;
(2) *The sequence* (2.2) *splits;*
(3) *The sequence* (2.3) *splits;*
(4) $\Omega_{R^\#}(N) \in \Sigma_k$;
(5) y^k *annihilates* $\operatorname{Ext}^1_{R^\#}(N, \Omega_{R^\#}(N))$. $\square$

Remark 2.9. In [1] Auslander and Solberg develop relative homological algebra from the perspective of sub-functors of $\operatorname{Ext}^1(-,-)$. For any $k \geqslant 1$, $F_k(-,-) := y^k \operatorname{Ext}^1_{R^\#}(-,-)$ is such a sub-functor on $\operatorname{MCM}(R^\#)$. The corresponding category of relatively F_k-projectives consists of all N such that $F_k(N,-) = y^k \operatorname{Ext}^1_{R^\#}(N,-) = 0$, which coincides with Σ_k. Likewise, the category of relatively F_k-injectives, consisting of all N such that $F_k(-,N) = y^k \operatorname{Ext}^1_{R^\#}(-,N) = 0$, also coincides with Σ_k, as Σ_k is closed under syzygies and $\Omega^2_{R^\#}$ is isomorphic to the identity. The fact that Σ_k is functorially finite translates to the existence of enough relative F_k-projectives (and injectives). Moreover, as the relative projectives and injectives coincide, for each $k \geqslant 1$ we see that the (relative) exact structure of $\operatorname{MCM}(R^\#)$ with Σ_k as the (relative) projectives is a (relative) Frobenius category structure.

Proposition 2.10. *Assume in addition that S is complete and that N is indecomposable. Then* (2.2) *is a minimal right approximation of N as long as it is not split.*

Proof. Since MCM $R^\#$-modules satisfy the Krull–Remak–Schmidt uniqueness property for direct-sum decompositions, a right approximation $q\colon X \longrightarrow M$ is non-minimal if and only if it vanishes on a direct summand of X. When N is indecomposable, the kernel of the approximation $\Omega_{R^\#}(N)$ is indecomposable as well, and it cannot contain a direct summand of X other than itself, in which case the sequence splits. $\square$

It does not seem possible in general to identify the first k so that (2.2) splits. By Proposition 2.3 and the long exact sequence in Ext, it coincides with the least k so that multiplication y^k on N factors through a free $R^\#$-module. Equivalently, multiplication by y^k is in the image of the natural map $N^* \otimes_{R^\#} N \longrightarrow \operatorname{End}_{R^\#}(N)$. In the special case $n = 2$, it is known (see for example [12, Proposition 8.30]) that for an indecomposable MCM R-module M, one has $\Omega_{R^\#}(M)$ indecomposable if and only if $M \cong \Omega_R(M)$. As far as we are aware, no criterion of this form is known for $n \geqslant 3$.

3. Branched covers of simple singularities

In this section we consider the case where $R = S/(f)$ is a hypersurface of *finite CM representation type* (of dimension $d \geqslant 1$). If S is a power series ring over an algebraically closed field of characteristic zero, this is equivalent to R being a *simple singularity,* that is, $R \cong k[\![x, y, z_2, \dots, z_d]\!]/(f)$, where f is one of the following polynomials:

(A_n): $x^2 + y^{n+1} + z_2^2 + \cdots + z_d^2$, $n \geqslant 1$
(D_n): $x^2y + y^{n-1} + z_2^2 + \cdots + z_d^2$, $n \geqslant 4$
(E_6): $x^3 + y^4 + z_2^2 + \cdots + z_d^2$
(E_7): $x^3 + xy^3 + z_2^2 + \cdots + z_d^2$
(E_8): $x^3 + y^5 + z_2^2 + \cdots + z_d^2$

We point out that each of these is an iterated double (2-fold) branched cover of the one-dimensional simple singularity; Knörrer's Theorem 1.1 allows one to show that these hypersurface rings have finite CM representation type by reducing to the 1-dimensional case. For $n \geqslant 3$, adding a summand of the form z^n to a simple singularity gives a non-simple singularity (except in type A_1 and some isolated examples for small n).

Since R has finite CM representation type, the category $\Sigma_1 \subset \operatorname{MCM}(R^\#)$ has only finitely many indecomposable objects up to isomorphism. We let M be the direct sum of one copy of each indecomposable module in $\operatorname{MCM} R$ up to isomorphism. Then $\operatorname{add} M = \operatorname{MCM} R$, and we say that such an M is a (basic) *representation generator* for $\operatorname{MCM} R$. Setting $\widetilde{M} := \Omega_{R^\#}(M)$, we have $\operatorname{add} \widetilde{M} = \Sigma_1$, so that $\widetilde{M}$ is a (not necessarily basic) representation generator for Σ_1. We note that $\widetilde{M}$ has a direct summand isomorphic to $R^\#$.

We set $\Lambda_0 = \operatorname{End}_R(M)$ and $\Lambda = \operatorname{End}_{R^\#}(\widetilde{M})$. Recall the following result due independently to Iyama [9, 5.4.1], Leuschke [11, Prop. 8], and Quarles [16, Theorem 3.5.3].

Proposition 3.1. *The endomorphism ring Λ_0 has global dimension at most* $\max\{2, d\}$, *with equality holding if $d \geqslant 2$.* $\square$

The main result of this section is that Λ can be thought of as a "non-commutative hypersurface", in that every finitely generated Λ-module has a projective resolution which is eventually periodic of period at most 2. Furthermore Λ is Iwanaga–Gorenstein, and we identify the relative Gorenstein-projectives.

Definition 3.2. Let A be a ring and M a left A-module. A *complete resolution* of M is an exact sequence of projective A-modules

$$\mathbb{P}_\bullet = \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \longrightarrow \cdots$$

such that $\operatorname{Hom}_A(\mathbb{P}_\bullet, A)$ is exact and $M \cong \operatorname{im}(d_0)$. A module M admitting a complete resolution is said to be *Gorenstein projective* or *totally reflexive*. We will write $\operatorname{GP}(A)$ for the full subcategory of A-mod consisting of Gorenstein projective modules.

We regard $\widetilde{M}$ as a left $R^\#$-module and a right Λ-module, so that we have functors

$$\operatorname{Hom}_{R^\#}(\widetilde{M}, -)\colon R^\#\text{-Mod} \longrightarrow \Lambda\text{-Mod}$$

and

$$\operatorname{Hom}_{R^\#}(-, \widetilde{M})\colon R^\#\text{-Mod} \longrightarrow \text{Mod-}\Lambda$$

which induce equivalences from $\operatorname{add}\widetilde{M}$ to the subcategories of finitely generated projective left (respectively, right) Λ-modules.

Lemma 3.3. *Restricted to* $\operatorname{MCM}(R^\#)$*, we have a commutative diagram of functors up to natural isomorphism.*

$$\begin{tikzcd}
\operatorname{MCM}(R^\#) \arrow[r, "\operatorname{Hom}_{R^\#}(\widetilde{M},-)"] \arrow[dr, "\operatorname{Hom}_{R^\#}(-,\widetilde{M})"'] & \Lambda\text{-}\operatorname{mod} \arrow[d, "\operatorname{Hom}_\Lambda(-,\Lambda)"] \\
& \operatorname{mod}\text{-}\Lambda
\end{tikzcd}$$

Proof. By Theorem 2.7, any MCM $R^\#$-module N has a right Σ_1-approximation

$$0 \longrightarrow \Omega_{R^\#}(N) \longrightarrow T \longrightarrow N \longrightarrow 0\,, \tag{3.1}$$

where in fact $T = \Omega_{R^\#}(N/yN)$. Furthermore $\Omega_{R^\#}(N)$ has a right Σ_1-approximation

$$0 \longrightarrow N \longrightarrow \Omega_{R^\#}(T) \longrightarrow \Omega_{R^\#}(N) \longrightarrow 0\,. \tag{3.2}$$

Since these sequences are Σ_1-exact and Σ_1-coexact, we obtain exact sequences of Λ-modules

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$$0 \longrightarrow \mathrm{Hom}_{R^\#}(\widetilde{M}, N) \longrightarrow \mathrm{Hom}_{R^\#}(\widetilde{M}, \Omega_{R^\#}(T)) \longrightarrow \mathrm{Hom}_{R^\#}(\widetilde{M}, T) \longrightarrow \mathrm{Hom}_{R^\#}(\widetilde{M}, N) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}_{R^\#}(N, \widetilde{M}) \longrightarrow \mathrm{Hom}_{R^\#}(T, \widetilde{M}) \longrightarrow \mathrm{Hom}_{R^\#}(\Omega_{R^\#}(T), \widetilde{M}) \longrightarrow \mathrm{Hom}_{R^\#}(N, \widetilde{M}) \longrightarrow 0\,,$$

which yield projective presentations for the Λ-modules $\mathrm{Hom}_{R^\#}(\widetilde{M}, N)$ and $\mathrm{Hom}_{R^\#}(N, \widetilde{M})$ respectively. Thus these modules are finitely presented over Λ. We now apply $\mathrm{Hom}_\Lambda(-, \Lambda)$ to the first of these and compare with the second to obtain an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_\Lambda(\mathrm{Hom}_{R^\#}(\widetilde{M}, N), \Lambda) \arrow[r] & \mathrm{Hom}_\Lambda(\mathrm{Hom}_{R^\#}(\widetilde{M}, T), \Lambda) \arrow[r] & \mathrm{Hom}_\Lambda(\mathrm{Hom}_{R^\#}(\widetilde{M}, \Omega_{R^\#}(T)), \Lambda) \\
0 \arrow[r] & \mathrm{Hom}_{R^\#}(N, \widetilde{M}) \arrow[r] \arrow[u, dashed] & \mathrm{Hom}_{R^\#}(T, \widetilde{M}) \arrow[r] \arrow[u, "\cong"] & \mathrm{Hom}_{R^\#}(\Omega_{R^\#}(T), \widetilde{M}) \arrow[u, "\cong"]
\end{tikzcd}$$

Here the vertical maps are induced by the functor $\mathrm{Hom}_{R^\#}(\widetilde{M}, -)$, and the two on the right are isomorphisms since $T, \Omega_{R^\#}(T) \in \operatorname{add} \widetilde{M}$. It follows that the induced map on the left is also an isomorphism. $\square$

Proposition 3.4. *Let Z be a left (resp. right) Λ-module of the form $\mathrm{Hom}_{R^\#}(\widetilde{M}, N)$ (resp. $\mathrm{Hom}_{R^\#}(N, \widetilde{M})$), where N is a MCM $R^\#$-module. Then Z has a complete Λ-resolution which is periodic of period at most 2.*

Proof. As in the preceding proof we have exact sequences (3.1) and (3.2), which we can splice together to form a doubly-infinite 2-periodic exact sequence of $R^\#$-modules

$$\mathbb{T}\colon\ \cdots \longrightarrow \Omega_{R^\#}(T) \longrightarrow T \longrightarrow \Omega_{R^\#}(T) \longrightarrow T \longrightarrow \cdots\,. \tag{3.3}$$

Since $\widetilde{M}$ is in Σ_1, the induced sequences $\mathrm{Hom}_{R^\#}(\widetilde{M}, \mathbb{T})$ and $\mathrm{Hom}_{R^\#}(\mathbb{T}, \widetilde{M})$ are still exact, and since both T and $\Omega_{R^\#}(T)$ are in $\Sigma_1 = \operatorname{add} \widetilde{M}$, each module in the sequence is a projective Λ-module. Thus $\mathrm{Hom}_{R^\#}(\widetilde{M}, \mathbb{T})$ (respectively, $\mathrm{Hom}_{R^\#}(\mathbb{T}, \widetilde{M})$) is a projective resolution (and co-resolution) of $Z = \mathrm{Hom}_{R^\#}(\widetilde{M}, N)$ (respectively, of $Z' = \mathrm{Hom}_{R^\#}(N, \widetilde{M})$).

It remains to show that $\mathrm{Hom}_{R^\#}(\widetilde{M}, \mathbb{T})$ and $\mathrm{Hom}_{R^\#}(\mathbb{T}, \widetilde{M})$ remain exact upon dualizing into Λ, and thus are complete resolutions. However, by Lemma 3.3 we know that these resolutions are dual to each other under $\mathrm{Hom}_\Lambda(-, \Lambda)$, and the result follows since they are both exact. $\square$

Remark 3.5. In fact the resolution we build is something stronger than periodic of period 2 – it is built out of a single map $f\colon \Omega(T) \longrightarrow T$ and its syzygy $\Omega(f)\colon T \longrightarrow \Omega(T)$.

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Lemma 3.6. *Set $m = \max\{2, d+1\}$. Let X be an arbitrary finitely generated left (resp. right) Λ-module. Then there is a MCM $R^\#$-module N such that $\Omega^m_\Lambda(X) \cong \operatorname{Hom}_{R^\#}(\widetilde{M}, N)$ (resp. $\Omega^m_\Lambda(X) \cong \operatorname{Hom}_{R^\#}(N, \widetilde{M})$).*

Proof. First assume that X is a left Λ-module. Set $Z = \Omega^m_\Lambda(X)$, and let

$$0 \longrightarrow Z \longrightarrow P_{m-1} \xrightarrow{\partial_{m-1}} P_{m-2} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \longrightarrow X \longrightarrow 0$$

be the first $m-1$ steps of a resolution of X by finitely generated projective Λ-modules. By the Yoneda Lemma, each $\partial_j \colon P_j \longrightarrow P_{j-1}$ is of the form $\operatorname{Hom}_{R^\#}(\widetilde{M}, f_j)$ for some $R^\#$-linear $f_j \colon M_j \longrightarrow M_{j-1}$, where each $M_j \in \operatorname{add} \widetilde{M}$. (Notice that since $m \geqslant 2$, there is indeed at least one ∂_j.) We thus obtain a sequence of $R^\#$-modules and homomorphisms

$$0 \longrightarrow \ker f_{m-1} \longrightarrow M_{m-1} \xrightarrow{f_{m-1}} M_{m-2} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{f_1} M_0 \,. \tag{3.4}$$

Since $\widetilde{M}$ has an $R^\#$-free summand and the result of applying $\operatorname{Hom}_{R^\#}(\widetilde{M}, -)$ to (3.4) is exact, in fact (3.4) must be exact. In particular $N := \ker f_{m-1}$ has depth at least $m \geqslant d+1$, and is hence a MCM $R^\#$-module. Furthermore, we have $\operatorname{Hom}_{R^\#}(\widetilde{M}, N) \cong Z$ by left-exactness of Hom.

Similarly, if X is a right Λ-module, we can take a projective resolution $\mathbb{P} \cong \operatorname{Hom}_{R^\#}(\mathbb{M}, \widetilde{M})$ of X, where $\mathbb{M}$ denotes a sequence $M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \longrightarrow M_{m-2} \xrightarrow{f_{m-1}} M_{m-1} \longrightarrow \cdots$ in $\operatorname{add} \widetilde{M}$. Since $R^\#$ is a summand of $\widetilde{M}$ and $\mathbb{P}$ is exact, we have an exact sequence

$$0 \longrightarrow K \longrightarrow M^*_{m-1} \xrightarrow{f^*_{m-1}} M^*_{m-2} \longrightarrow \cdots \longrightarrow M^*_1 \xrightarrow{f^*_1} M^*_0 \,,$$

where we set $K := \ker(f^*_{m-1})$, writing $(-)^*$ for the exact duality $\operatorname{Hom}_{R^\#}(-, R^\#)$. Since each M^*_i is a MCM $R^\#$-module, K has depth at least $m \geqslant d+1$, and is hence a MCM $R^\#$-module. Taking the $R^\#$-dual now shows that $K^* \cong \operatorname{cok}(f_{m-1})$, and the left-exactness of $\operatorname{Hom}_{R^\#}(-, \widetilde{M})$ yields that $\operatorname{Hom}_{R^\#}(K^*, \widetilde{M}) \cong \ker \operatorname{Hom}_{R^\#}(f_{m-1}, \widetilde{M}) \cong \Omega^m_\Lambda(X)$. $\square$

Theorem 3.7. *Let $R = S/(f)$ be a hypersurface of finite CM representation type, with representation generator M. Let $R^\# = S[y]/(f + y^n)$ for some $n \geqslant 1$, and set $\Lambda = \operatorname{End}_{R^\#}(\Omega_{R^\#}(M))$. Then every finitely generated left (resp. right) Λ-module has a projective resolution which is eventually periodic of period at most 2.*

Proof. Let X be a finitely generated left (resp. right) Λ-module. By Lemma 3.6, $\Omega^m_\Lambda(X)$ is of the form $\operatorname{Hom}_{R^\#}(\widetilde{M}, N)$ (resp. $\operatorname{Hom}_{R^\#}(N, \widetilde{M})$) for some MCM $R^\#$-module N, where $m = \max\{2, \dim R+1\}$. It then follows from Proposition 3.4 that $\Omega^m_\Lambda(X)$ has a 2-periodic complete resolution. Splicing the left-hand half of this complete resolution together with

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the first $m-1$ steps of the projective resolution of X, we obtain an eventually periodic resolution. $\square$

Remark 3.8. While we think of the ring Λ as a "non-commutative hypersurface" in a homological sense, on the basis of Theorem 3.7, we should emphasize that as far as we know there does not exist a ring Γ of finite global dimension such that $\Lambda \cong \Gamma/(g)$ for some element g.

Remark 3.9. It seems likely that some version of many of the results in this section hold more generally (that is, without the assumption that R is a simple singularity) if one uses the formalism of "rings with several objects" as in, for example, [6]. We don't pursue this direction.

Recall that a Noetherian ring A is said to be Iwanaga–Gorenstein of dimension at most d if $\operatorname{id}_A A \leqslant d$ and $\operatorname{id} A_A \leqslant d$.

Proposition 3.10. *The ring* $\Lambda = \operatorname{End}_{R^\#}(\Omega_{R^\#}(M))$ *constructed above is Iwanaga–Gorenstein of dimension at most* $m = \max\{2, d+1\}$.

Proof. Set $m = \max\{2, d+1\}$ and observe that for any finitely generated left (resp. right) Λ-module X, we have

$$\operatorname{Ext}_\Lambda^{m+1}(X, \Lambda) \cong \operatorname{Ext}_\Lambda^1(\Omega_\Lambda^m(X), \Lambda) \cong 0\,,$$

since $\Omega_\Lambda^m(X)$ is Gorenstein projective by Lemma 3.6 and Proposition 3.4, and Gorenstein projectives are Ext-orthogonal to Λ by definition. It follows that $\operatorname{id}_\Lambda \Lambda \leqslant m$ and $\operatorname{id} \Lambda_\Lambda \leqslant m$. $\square$

Theorem 3.11. *The functor* $F := \operatorname{Hom}_{R^\#}(\widetilde{M}, -)$ *induces an equivalence of categories from* $\operatorname{MCM}(R^\#)$ *to* $\operatorname{GP}(\Lambda)$. *In particular, a left* Λ*-module* X *is Gorenstein projective if and only if* $X \cong \operatorname{Hom}_{R^\#}(\widetilde{M}, N)$ *for a MCM* $R^\#$*-module* N.

Proof. We begin with the second claim, which establishes denseness of F. By Proposition 3.4 we know that every Λ-module of the form $\operatorname{Hom}_{R^\#}(\widetilde{M}, N)$ with N in $\operatorname{MCM}(R^\#)$ is Gorenstein projective. Conversely, suppose that X belongs to $\operatorname{GP}(\Lambda)$. By Lemma 3.6, we see that $\Omega_\Lambda^m(X) \cong \operatorname{Hom}_{R^\#}(\widetilde{M}, N)$ for some N in $\operatorname{MCM}(R^\#)$, and thus by Proposition 3.4, we know $\Omega_\Lambda^m(X)$ has a complete resolution of the form $\operatorname{Hom}_{R^\#}(\widetilde{M}, \mathbb{T})$ for an exact sequence $\mathbb{T}$ as in (3.3). Here, as there, $m = \max\{2, d+1\}$. Comparing this complete resolution to that of X we see that, up to projective direct summands, $X \cong \operatorname{Hom}_{R^\#}(\widetilde{M}, N)$ or $X \cong \operatorname{Hom}_{R^\#}(\widetilde{M}, \Omega_{R^\#} N)$ depending on whether m is even or odd, respectively.

The functor F is easily seen to be faithful since $R^\#$ is a direct summand of $\widetilde{M}$. To see that it is full, consider a map $g\colon F(N) \longrightarrow F(N')$ for N, N' in $\operatorname{MCM}(R^\#)$. Then g can be lifted to a map of projective presentations as in the diagram,

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$$\begin{tikzcd}
F(M_1) \arrow[r, "F(f_1)"] \arrow[d, "g_1"] & F(M_0) \arrow[r, "F(f_0)"] \arrow[d, "g_0"] & F(N) \arrow[r] \arrow[d, "g"] & 0 \\
F(M_1') \arrow[r, "F(f_1')"] & F(M_0') \arrow[r, "F(f_0')"] & F(N') \arrow[r] & 0
\end{tikzcd}$$

with $M_i, M_i' \in \operatorname{add} \widetilde{M}$. Since F induces an equivalence from $\operatorname{add} \widetilde{M}$ to $\operatorname{add}(\Lambda)$, we can write $g_i = F(h_i)$ for suitable maps h_i (for $i = 1, 2$), and we obtain a commutative exact diagram in $\mathrm{MCM}(R^\#)$

$$\begin{tikzcd}
M_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & M_0 \arrow[r, "f_0"] \arrow[d, "h_0"] & N \arrow[r] \arrow[d, dashed, "h"] & 0 \\
M_1' \arrow[r, "f_1'"] & M_0' \arrow[r, "f_0'"] & N' \arrow[r] & 0
\end{tikzcd}$$

yielding a map $h \colon N \longrightarrow N'$. It then follows that $g = F(h)$. $\square$

Remark 3.12. The preceding Proposition and Theorem can also be deduced from recent work of Iyama, Kalck, Wemyss and Yang [8]. To see this, recall from Remark 2.9 that the category $\mathrm{MCM}(R^\#)$ has a (relative) Frobenius category structure obtained by taking $\Sigma_1 = \operatorname{add} \widetilde{M}$ as the subcategory of (relative) projective objects, and Σ_1-exact sequences as the exact sequences. Since the functor categories mod-$\mathrm{MCM}(R^\#)$ and mod-$\mathrm{MCM}(R^\#)^{\mathrm{op}}$ have global dimension at most $m = \max\{2, d+1\}$ by [6, Theorem 4.10], Theorem 2.8 of [8] applies. Part (1) of that theorem corresponds to the Iwanaga–Gorenstein property of our $\Lambda = \operatorname{End}_{R^\#}(\widetilde{M})$, while part (2) gives the equivalence between $\mathrm{MCM}(R^\#)$ and $\mathrm{GP}(\Lambda)$. Of course, it also follows from this realization of a Frobenius structure, that we have an equivalence of triangulated categories induced by

$$\operatorname{Hom}_{R^\#}(\widetilde{M}, -) \colon\ \mathrm{MCM}(R^\#)/[\Sigma_1] \approx \underline{\mathrm{GP}}(\Lambda)\,,$$

where $[\Sigma_1]$ denotes the ideal of morphisms that factor through an object in the subcategory Σ_1 and we write $\underline{\mathrm{GP}}(\Lambda)$ for the stable category of Gorenstein projective Λ-modules.

4. Generation of MCM module categories

When R is a Gorenstein local ring, $\mathrm{MCM}(R)$ is a Frobenius category, and thus the stable category $\underline{\mathrm{MCM}}(R)$, whose objects are the same as $\mathrm{MCM}(R)$ and whose Hom-sets are obtained by killing those morphisms factoring through a free R-module, is triangulated with suspension given by the co-syzygy functor Ω^{-1} [3]. This stable category is also equivalent [15] to the *singularity category* $D_{\mathrm{sg}}(R)$, that is, the Verdier quotient of

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the bounded derived category $D^b(\operatorname{mod} R)$ by the subcategory of perfect complexes. In this section we investigate the dimension of this triangulated category, in the sense of Rouquier, when R is an isolated hypersurface singularity. In this setting, Ballard, Favero and Katzarkov have found a general upper bound for the dimension of $\underline{\mathrm{MCM}}(R)$, showing in particular that it is always finite [2]. We focus on a special class of hypersurfaces, where we improve upon this upper bound.

We begin by reviewing the definition of the dimension of a triangulated category introduced by Rouquier [17]. Let $\mathcal{T}$ be a triangulated category and $\mathcal{I}$ a full subcategory of $\mathcal{T}$. (For ease of exposition, we will assume all full subcategories are closed under isomorphisms.) We let $\langle \mathcal{I} \rangle$ denote the smallest full subcategory of $\mathcal{T}$ that contains $\mathcal{I}$ and is closed under isomorphisms, direct summands, finite direct sums and shifts. For two full subcategories $\mathcal{I}_1$ and $\mathcal{I}_2$ we also write $\mathcal{I}_1 * \mathcal{I}_2$ for the full subcategory of all Y for which $\mathcal{T}$ contains a distinguished triangle $X_1 \longrightarrow Y \longrightarrow X_2 \longrightarrow X_1[1]$ with $X_i \in \mathcal{I}_i$, and we further set $\mathcal{I}_1 \diamond \mathcal{I}_2 = \langle \mathcal{I}_1 * \mathcal{I}_2 \rangle$. Observe that if $\mathcal{T} = \underline{\mathrm{MCM}}(R)$, then $Y \in \mathcal{I}_1 * \mathcal{I}_2$ if and only if there is a short exact sequence $0 \longrightarrow X_1 \longrightarrow Y \oplus F \longrightarrow X_2 \longrightarrow 0$ in $\mathrm{MCM}(R)$ with $X_i \in \mathcal{I}_i$ for $i = 1, 2$ and F free. Following the conventions of [2], we inductively define $\langle \mathcal{I} \rangle_0 = \langle \mathcal{I} \rangle$ and $\langle \mathcal{I} \rangle_n = \langle \mathcal{I} \rangle_{n-1} \diamond \langle \mathcal{I} \rangle$ for $n \geqslant 1$.

Definition 4.1 *(Rouquier).* The *dimension* of $\mathcal{T}$ is the smallest integer n such that $\mathcal{T} = \langle X \rangle_n$ for some object X in $\mathcal{T}$, or else ∞ if no such n exists.

If $R = k[\![x_0, \dots, x_d]\!]/(f)$ is a complete isolated hypersurface singularity over an algebraically closed field k of characteristic zero, Ballard, Favero and Katzarkov bound the dimension of $\underline{\mathrm{MCM}}(R)$ in terms of the Tjurina algebra $k[\![x_0, \dots, x_d]\!]/\Delta_f$, where Δ_f is the ideal in $k[\![x_0, \dots, x_d]\!]$ generated by the partial derivatives of f with respect to the x_i. Recall that this algebra is Artinian and local.

Theorem 4.2 *([2, Proposition 4.11]). Let R be a d-dimensional isolated hypersurface singularity as above, with ℓ denoting the Loewy length of the Tjurina algebra of R. Then $\underline{\mathrm{MCM}}(R) = \langle \Omega^d(k) \rangle_{2\ell-1}$, and in particular* $\dim(\underline{\mathrm{MCM}}(R)) \leqslant 2\ell - 1$.

In particular, if R is defined by the polynomial $f = x_0^{a_0} + \cdots + x_d^{a_d}$, where each a_i is at least 2 and not divisible by $\operatorname{char}(k)$, then the Tjurina algebra of R is isomorphic to $k[x_0, \dots, x_d]/(x_0^{a_0-1}, \dots, x_d^{a_d-1})$, and it is easy to see that its Loewy length is

$$\ell = \sum_{i=0}^{d} (a_i - 2) + 1 = \sum_{i=0}^{d} a_i - 2d - 1\,.$$

In this case, we have $\dim(\underline{\mathrm{MCM}}(R)) \leqslant 2 \sum_{i=0}^{d} a_i - 4d - 3$.

We now return to the context explored in the previous section. Thus assume that R is an isolated hypersurface singularity given by a non-zero power series f in $x_0, \dots, x_d$ and set $R^\# = k[\![x_0, \dots, x_d, y]\!]/(f + y^n)$. For $1 \leqslant i \leqslant n$, we set

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$$\Sigma_i = \operatorname{add}\left\{\Omega_{R^\#}(M) \,\middle|\, M \in \operatorname{MCM}(R^\#/(y^i))\right\} ,$$

and continue to write Σ_i for its image in the stable category. Note that Σ_i is closed under isomorphisms, finite direct sums and direct summands by definition, and is closed under shifts by Corollary 2.8.

The following lemma, which is a slight generalization of [7, Lemma 2.4], will be used repeatedly in what follows. As the lemma can be stated over any ring Γ, we recall that syzygies are only defined up to projective summands. Thus for each left Γ-module M we fix a projective resolution $(F_i^M, \partial_i^M)_{i\geqslant 0}$ of M and define $\Omega^i_\Gamma(M) = \ker \partial^M_{i-1}$. We say that two Γ-modules X and Y are *projectively equivalent* if $X \oplus P \cong Y \oplus P'$ for some projective Γ-modules P and P', and we write $X \sim Y$ in this case.

Lemma 4.3. *Let Γ be any ring, let I be a proper two-sided ideal of Γ such that $\overline{\Gamma} = \Gamma/I$ has finite projective dimension $r \geqslant 1$ as a left Γ-module. For any left $\overline{\Gamma}$-module X we have*

$$\Omega^{r+1}_\Gamma(X) \sim \Omega^r_\Gamma(\Omega_{\overline{\Gamma}}(X)) .$$

More generally, for any $m \geqslant 1$, we have

$$\Omega^{r+m}_\Gamma(X) \sim \Omega^r_\Gamma(\Omega^m_{\overline{\Gamma}}(X)) .$$

Proof. Fix a projective resolution $\cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} X \longrightarrow 0$ so that $\Omega^i_\Gamma(X) = \ker \partial_{i-1}$ for each $i \geqslant 1$. Tensoring down to $\overline{\Gamma}$, we obtain an exact commutative diagram of Γ-modules

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & IF_0 \arrow[r, equal] \arrow[d] & IF_0 \arrow[d] & & \\
0 \arrow[r] & \Omega_\Gamma X \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & F_0/IF_0 \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where $K \oplus Q_1 \cong \Omega_{\overline{\Gamma}} X \oplus Q_2$ for projective $\overline{\Gamma}$-modules Q_1, Q_2 by Schanuel's Lemma. Since $\operatorname{pd}({}_\Gamma\overline{\Gamma}) = r$, we see that $\Omega^r_\Gamma(Q_i)$ is a projective Γ-module for each i. Thus, $\Omega^r_\Gamma K$ and $\Omega^r_\Gamma(\Omega_{\overline{\Gamma}} X)$ will be projectively equivalent over Γ.

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We then have a second diagram as below.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & IF_0 \arrow[d] & \\
0 \arrow[r] & \Omega^2_\Gamma X \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d, equal] & \Omega_\Gamma X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
 & IF_0 \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Considering the vertical sequence on the left, we can use the horseshoe lemma to obtain a projective resolution of L and a short exact sequence of syzygies

$$0 \longrightarrow \Omega^{r+1}_\Gamma X \longrightarrow \Omega^{r-1}_\Gamma L \oplus P \longrightarrow \Omega^{r-1}_\Gamma(IF_0) \oplus P' \longrightarrow 0$$

for projective Γ-modules P and P'. Since $IF_0 \in \operatorname{add}(I)$ and ${}_\Gamma I$ has projective dimension $r-1$, we see that $\Omega^{r-1}_\Gamma(IF_0)$ is projective and this sequence splits. It follows that $\Omega^{r+1}_\Gamma X$ is projectively equivalent to $\Omega^{r-1}_\Gamma L$. Since $L \sim \Omega_\Gamma(K)$, we see that $\Omega^{r+1}_\Gamma X$ is projectively equivalent to $\Omega^r_\Gamma K$ and thus to $\Omega^r_\Gamma(\Omega_{\overline{\Gamma}} X)$.

The second claim now follows via induction on $m \geqslant 1$. $\square$

In particular, when the ideal I is generated by a central regular sequence of length r, we have $\operatorname{pd}({}_\Gamma\Gamma/I) = r$ and thus the preceding lemma applies in this situation. Furthermore, for such ideals I in a commutative ring Γ, the Eagon–Northcott resolution shows that $\operatorname{pd}({}_\Gamma\Gamma/I^j) = r$ for each $j \geqslant 1$ [5], and thus the lemma also applies for the ideal I^j, as will be needed a bit later on.

Proposition 4.4. *We have $\Sigma_k = \Sigma_{k-j} \diamond \Sigma_j$ for every $1 \leqslant j < k \leqslant n$. In particular,* $\operatorname{MCM}(R^\#) = \Sigma_{n-1} = \langle \Sigma_1 \rangle_{n-2}$.

Proof. First let $M \in \Sigma_k$, say $M \oplus M' = \Omega_{R^\#}(X)$ with X an $R^\#$-module annihilated by y^k. Then we have a short exact sequence

$$0 \longrightarrow y^j X \longrightarrow X \longrightarrow X/y^j X \longrightarrow 0\,, \tag{4.1}$$

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in which the leftmost, resp. rightmost, term is annihilated by y^{k-j}, resp. y^j. Observe, however, that neither of these need be MCM over $R^\#/(y^{k-j})$, resp. $R^\#/(y^j)$. Fix an odd integer $m > d$, and apply $\Omega^m_{R^\#}$ to (4.1), obtaining

$$0 \longrightarrow \Omega^m_{R^\#}(y^jX) \longrightarrow \Omega^m_{R^\#}(X) \oplus F \longrightarrow \Omega^m_{R^\#}(X/y^jX) \longrightarrow 0 \tag{4.2}$$

for some free $R^\#$-module F. Since m is odd, the middle term is isomorphic to $\Omega^1_{R^\#}(X) \oplus F \cong M \oplus M' \oplus F$. By Lemma 4.3, since powers of y are regular in $R^\#$, we may rewrite the outer terms as the first syzygies, over $R^\#$, of $\Omega^{m-1}_{R^\#/(y^{k-j})}(y^jX)$ and $\Omega^{m-1}_{R^\#/(y^j)}(X/y^jX)$, respectively. For m large enough, these are both MCM over the appropriate quotient of $R^\#$, and it follows that $\Omega^m_{R^\#}(y^jX) \in \Sigma_{k-j}$ and $\Omega^m_{R^\#}(X/y^jX) \in \Sigma_j$. Hence $M \in \Sigma_{k-j} \diamond \Sigma_j$.

For the other containment, it suffices to complete a diagram of the form

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \Omega_{R^\#}(X_1) \arrow[r] \arrow[d] & M \oplus M' \arrow[r] & \Omega_{R^\#}(X_2) \arrow[r] \arrow[d] & 0 \\
 & F_1 \arrow[d] & & F_2 \arrow[d] & \\
 & X_1 \arrow[d] & & X_2 \arrow[d] & \\
 & 0 & & 0 &
\end{tikzcd}$$

where $y^{k-j}X_1 = y^jX_2 = 0$ and F_1, F_2 are free $R^\#$-modules. Apply $\operatorname{Hom}_{R^\#}(-, F_1)$ to the top row, obtaining

$$\cdots \longrightarrow \operatorname{Hom}_{R^\#}(M \oplus M', F_1) \longrightarrow \operatorname{Hom}_{R^\#}(\Omega_{R^\#}(X_1), F_1) \longrightarrow \operatorname{Ext}^1_{R^\#}(\Omega_{R^\#}(X_2), F_1)\,.$$

Since $\Omega_{R^\#}(X_2)$ is MCM and F_1 is free, $\operatorname{Ext}^1_{R^\#}(\Omega_{R^\#}(X_2), F_1) = 0$. This implies that the map $\Omega_{R^\#}(X_1) \longrightarrow F_1$ extends to a map $M \oplus M' \longrightarrow F_1$, and we therefore obtain a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{R^\#}(X_1) \arrow[r] \arrow[d] & M \oplus M' \arrow[r] \arrow[d] & \Omega_{R^\#}(X_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \arrow[r] & F_1 \oplus F_2 \arrow[r] & F_2 \arrow[r] & 0\,.
\end{tikzcd}$$

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Set $X = \operatorname{cok}(M \oplus M' \longrightarrow F_1 \oplus F_2)$. Then of course $M \oplus M' = \Omega_{R\#}(X)$ and we have a short exact sequence $0 \longrightarrow X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow 0$. Since $y^{k-j}X_1 = 0$ and $y^j X_2 = 0$, we have $y^k X = 0$, as required. □

We now extend this result to iterated branched covers, as considered by O'Carroll and Popescu in [14]. Let S be a $d+1$-dimensional regular local ring, and set $R = S/(f)$ and

$$R_r = S[\![y_1, \dots, y_r]\!]/(f + y_1^{a_1} + \cdots y_r^{a_r})$$

for positive integers $a_i \geqslant 2$. Let I_r be the ideal $(y_1, \dots, y_r)R_r$, and define

$$\Sigma_j^r := \operatorname{add}\{\Omega_{R_r}^r(X) \mid X \in \operatorname{MCM}(R_r/I_r^j)\}\,.$$

Note that $y_1, \dots, y_r$ is a regular sequence on R_r, so that R_r/I_r^j is a CM ring of dimension d for any j (by, for example, Eagon–Northcott [5, Theorem 2]). It follows that any $X \in \operatorname{MCM}(R_r/I_r^j)$ has depth d as an R_r-module, and thus that $\Sigma_j^r \subseteq \operatorname{MCM}(R_r)$.

The argument outlined in Remark 2.4 has been extended by Takahashi [18] to recover the following result of O'Carroll and Popescu, which is a consequence of Theorem 2.2 in [14]:

Proposition 4.5. *Assume that each a_i is invertible in S. Then for any MCM R_r-module N, we have*

$$\Omega_{R_r}^r(N/(y_1^{a_1-1}, \dots, y_r^{a_r-1})N) \cong \bigoplus_{j=0}^{r} \Omega_{R_r}^j(N)^{\binom{r}{j}}\,. \quad \square$$

The necessary ingredients in proving Proposition 4.5 are (i) to observe that the Jacobian ideal $(y_1^{a_1-1}, \dots, y_r^{a_r-1})$ annihilates $\operatorname{Ext}_{R_r}^1(N, \Omega_{R_r}(N))$, and (ii) to induct over the number r of new variables by applying Lemma 4.3. We omit the details.

We set $m := \sum_{i=1}^r (a_i - 2) + 1$, so that $I_r^m \subseteq (y_1^{a_1-1}, \dots, y_r^{a_r-1})$. By Proposition 4.5, every MCM R_r-module N is a direct summand of $\Omega_{R_r}^r(N/(y_1^{a_1-1}, \dots, y_r^{a_r-1})N)$. Since $N/(y_1^{a_1-1}, \dots, y_r^{a_r-1})N$ is a MCM R_r/I_r^m-module, this yields $\Sigma_m^r = \operatorname{MCM}(R_r)$.

Theorem 4.6. *For each $1 \leqslant j < k$ we have $\Sigma_k^r = \Sigma_{k-j}^r \diamond \Sigma_j^r = \langle \Sigma_1^r \rangle_{k-1}$. In particular, $\operatorname{MCM}(R_r) = \Sigma_m^r = \langle \Sigma_1^r \rangle_{m-1}$.*

Proof. Step 1. We first show the inclusions $\Sigma_k^r \subseteq \Sigma_{k-j}^r \diamond \Sigma_j^r \subseteq \langle \Sigma_1^r \rangle_{k-1}$. The argument is similar to the first part of the proof of Proposition 4.4. Let $M \in \Sigma_k^r$, say $M \oplus M' = \Omega_{R_r}^r(X)$ with X an R_r-module annihilated by I_r^k. Then, for any even integer $n > d$, the short exact sequence $0 \longrightarrow I_r^j X \longrightarrow X \longrightarrow X/I_r^j X \longrightarrow 0$ induces a short exact sequence $0 \longrightarrow \Omega_{R_r}^{r+n}(I_r^j X) \longrightarrow M \oplus M' \oplus F \longrightarrow \Omega_{R_r}^{r+n}(X/I_r^j X) \longrightarrow 0$ for some free module F. By Lemma 4.3, $\Omega_{R_r}^{r+n}(X/I_r^j X)$ is projectively equivalent to $\Omega_{R_r}^r(\Omega_{R_r/I_r^j}^n(X/I_r^j X))$ with $\Omega_{R_r/I_r^j}^n(X/I_r^j X)$ MCM over R_r/I_r^j. Similarly, $\Omega_{R_r}^{r+n}(I_r^j X)$ is projectively equivalent to $\Omega^r_{R_r}(\Omega^n_{R_r/I_r^{k-j}}(I_r^j X))$ with $\Omega^n_{R_r/I_r^{k-j}}(I_r^j X)$ MCM over R_r/I_r^{k-j}. In particular, $M \in \Sigma^r_{k-j} \diamond \Sigma^r_j$. The second inclusion now follows by definition of $\langle \Sigma^r_1 \rangle_{k-1}$ and induction on k.

Step 2. We show that $\Omega_{R_r}(\Sigma_i^{r-1}) \subseteq \Sigma_i^r$ for each $r \geqslant 2$ and $i \geqslant 1$. Let $M \in \Sigma_i^{r-1}$, so that $M \oplus M' \cong \Omega^{r-1}_{R_{r-1}}(X)$ for some $M' \in \mathrm{MCM}(R_{r-1})$ and $X \in \mathrm{MCM}(R_{r-1}/I^i_{r-1})$. Then

$$\Omega_{R_r} M \oplus \Omega_{R_r} M' \cong \Omega_{R_r}(\Omega^{r-1}_{R_{r-1}}(X)) \cong \Omega^r_{R_r}(X)\,,$$

where the last isomorphism holds up to free summands by Lemma 4.3. Thus $\Omega_{R_r} M$ belongs to Σ_i^r since X may also be viewed as a MCM module over R_r/I_r^i.

Step 3. We show that $\Sigma_1^r = \Omega_{R_r}(\Sigma_1^{r-1})$ for each $r \geqslant 2$. Let $M \in \Sigma_1^r$. Then $M \oplus M' \cong \Omega^r_{R_r} X$ for a MCM R_r/I_r-module X. But $R_r/I_r \cong R \cong R_{r-1}/I_{r-1}$, so X is an R_{r-1}-module and $\Omega^{r-1}_{R_{r-1}} X \in \Sigma_1^{r-1}$. Thus, up to free summands, $M \oplus M' \cong \Omega_{R_r}(\Omega^{r-1}_{R_{r-1}} X) \in \Omega_{R_r}(\Sigma_1^{r-1})$ as required.

Step 4. For each $j \geqslant 1$, consider the following two statements:

$$\forall\ r \geqslant 2 \quad \Sigma_j^r = \Omega_{R_r}(\Sigma_j^{r-1})\,, \tag{4.3}$$

and

$$\forall\ r \geqslant 1 \quad \mathrm{add}(\Sigma_1^r * \Sigma_j^r) = \Sigma_{j+1}^r\,. \tag{4.4}$$

For any fixed $j \geqslant 1$, we show that (4.3) implies (4.4). To see this, we apply induction on r, noting that the $r = 1$ case of (4.4) is handled by Proposition 4.4. By Step 1, it suffices to show $\mathrm{add}(\Sigma_1^r * \Sigma_j^r) \subseteq \Sigma_{j+1}^r$. Let $r \geqslant 2$. For any $M \in \mathrm{add}(\Sigma_1^r * \Sigma_j^r)$, we have an extension $0 \longrightarrow N \longrightarrow M \oplus M' \longrightarrow N' \longrightarrow 0$ for $N \in \Sigma_1^r = \Omega_{R_r}(\Sigma_1^{r-1})$ and $N' \in \Sigma_j^r = \Omega_{R_r}(\Sigma_j^{r-1})$. Now, as in the second half of the proof of Proposition 4.4, we obtain $M \in \Omega_{R_r}(\Sigma_1^{r-1} * \Sigma_j^{r-1}) = \Omega_{R_r}(\Sigma_{j+1}^{r-1}) \subseteq \Sigma_{j+1}^r$, where we have used the inductive hypothesis and the result of Step 2.

Step 5. We now prove (4.3), and hence also (4.4), for all $j \geqslant 1$ by induction on j. For $j = 1$, (4.3) was established in Step 3. Now, assume (4.3) holds for some $j \geqslant 1$, and let $M \in \Sigma_{j+1}^r = \mathrm{add}(\Sigma_1^r * \Sigma_j^r)$. The argument in the previous step shows that $M \in \Omega_{R_r}(\Sigma_{j+1}^{r-1})$, thus establishing (4.3) for $j+1$ with the help of Step 2.

Step 6. We finally show that $\Sigma_j^r = \langle \Sigma_1^r \rangle_{j-1}$ for all $r, j \geqslant 1$. It then follows that $\Sigma_k^r = \Sigma_{k-j}^r \diamond \Sigma_j^r$ for all $1 \leqslant j < k$. By Step 1, it suffices to show that $\langle \Sigma_1^r \rangle_{j-1} \subseteq \Sigma_j^r$. For $j = 1$, this amounts to saying that Σ_1^r is closed under syzygies. For $r = 1$, we know that Σ_j^r is closed under syzygies from Corollary 2.8. By induction on r, using (4.3) and Lemma 4.3, we see that all Σ_j^r are in fact closed under syzygies. Now using induction on j, we have

$$\langle \Sigma_1^r \rangle_j = \langle \Sigma_1^r * \langle \Sigma_1^r \rangle_{j-1} \rangle = \langle \Sigma_1^r * \Sigma_j^r \rangle = \langle \Sigma_{j+1}^r \rangle = \Sigma_{j+1}^r\,. \quad \square$$

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Corollary 4.7. *With notation as above,* $\dim(\mathrm{MCM}(R_r)) \leqslant \sum_{i=1}^{r}(a_i - 2)$ *whenever* R *has finite CM-type.*

Remark 4.8. We emphasize again that the Corollary above requires $f \neq 0$. Indeed, it is known that the category of MCM modules over the A_∞ singularity $k[\![x,y]\!]/(y^2)$ has dimension 1 [4, Proposition 3.7].

Question 4.9. We have shown that $\mathrm{MCM}(R_r) = \Sigma_m^r = \langle \Sigma_1^r \rangle_{m-1}$ and $\langle \Sigma_1^r \rangle_{m-2} = \Sigma_{m-1}^r$. But we do not know if $m-1$ is the smallest number of steps in which Σ_1^r generates $\mathrm{MCM}(R_r)$, or equivalently whether Σ_{m-1}^r must be a proper subset of $\Sigma_m^r = \mathrm{MCM}(R_r)$. Properness here would follow from the existence of an MCM R_r-module M for which $\mathrm{Ext}^1_{R_r}(M,-)$ is not annihilated by $y := \prod_{i=1}^{r} y_i^{a_i-2}$. To see this, note that if such an M is a direct summand of some $\Omega^r_{R_r}(X)$, then $\mathrm{Ext}^1_{R_r}(M,-)$ is isomorphic to a direct summand of $\mathrm{Ext}^{r+1}_{R_r}(X,-)$, and the latter must not be killed by y. Since $y \in I_r^{m-1}$, we see that X is not annihilated by I_r^{m-1} and thus $M \notin \Sigma^r_{m-1}$.

5. Examples

We illustrate the constructions in this paper in a couple of examples. For computational purposes it is convenient to work with matrix factorizations, on which the necessary background can be found in Chapter 7 of Yoshino's book [19]. For a power series f contained in the maximal ideal of $S = k[\![\underline{x}]\!]$, we write $\mathrm{MF}_S(f)$ for the category of reduced matrix factorizations of f over R and recall that the functor $\mathrm{cok}\colon \mathrm{MF}_R(f) \longrightarrow \mathrm{MCM}(R/(f))$, sending a matrix factorization (φ, ψ) to $\mathrm{cok}(\varphi)$, induces equivalences of categories $\mathrm{MF}_R(f)/[(1,f)] \approx \mathrm{MCM}(R/(f))$ and $\mathrm{MF}_R(f)/[(1,f),(f,1)] \approx \underline{\mathrm{MCM}}(R/(f))$. We also write $\Omega(\varphi,\psi) = (\psi,\varphi)$.

We begin with a description of the functor $\Omega_{R^\#}\colon \mathrm{MCM}(R) \to \mathrm{MCM}(R^\#)$ in terms of matrix factorizations. On this level, it turns out that this functor is a special case of Yoshino's tensor product of matrix factorizations [20]. Let $R = k[\![x_0,\dots,x_d]\!]$ and $R' = k[\![y_0,\dots,y_{d'}]\!]$, and set $S = k[\![x_0,\dots,x_d,y_0,\dots,y_{d'}]\!]$. If $X = (\varphi,\psi)$ and $X' = (\varphi',\psi')$ are matrix factorizations of $f \in R$ and $g \in R'$ respectively, of sizes n and m, then Yoshino defines the matrix factorization

$$X \hat{\otimes} X' = \left(\begin{bmatrix} \varphi \otimes I_m & I_n \otimes \varphi' \\ -I_n \otimes \psi' & \psi \otimes I_m \end{bmatrix}, \begin{bmatrix} \psi \otimes I_m & -I_n \otimes \varphi' \\ I_n \otimes \psi' & \varphi \otimes I_m \end{bmatrix} \right) \in \mathrm{MF}_S(f+g)\,.$$

For a fixed X', Yoshino shows that $-\hat{\otimes} X'\colon\ \mathrm{MF}_R(f) \longrightarrow \mathrm{MF}_S(f+g)$ is an exact functor that preserves trivial matrix factorizations, and behaves well with respect to syzygies. In particular, $X \hat{\otimes} \Omega X' \cong \Omega(X \hat{\otimes} X') \cong \Omega X \hat{\otimes} X'$. In addition, if X' is reduced, $-\hat{\otimes} X'$ is a faithful functor (although, it is typically very far from being full). We also have a reduction functor $- \otimes_S S/(\underline{y})\colon\ \mathrm{MF}_S(f+g) \longrightarrow \mathrm{MF}_R(f)$, which sends (Φ,Ψ) to $(\Phi \otimes_S S/(\underline{y}),\ \Psi \otimes_S S/(\underline{y}))$.

We now specialize to the setting considered in this paper: $R' = k[\![y]\!], g = y^n$ and $Y = (y, y^{n-1})$. Then the functor $-\hat{\otimes}Y$ sends an $m \times m$ matrix factorization (φ, ψ) of f over R to the $2m \times 2m$ matrix factorization

$$(\varphi,\psi)\hat{\otimes}(y,y^{n-1}) = \left(\begin{bmatrix} \varphi & yI_m \\ -y^{n-1}I_m & \psi \end{bmatrix}, \begin{bmatrix} \psi & -yI_m \\ y^{n-1}I_m & \varphi \end{bmatrix} \right).$$

Proposition 5.1. *Let $R = k[\![x_0, \dots, x_d]\!]/(f)$ be an isolated hypersurface singularity and set $R^\# = k[\![x_0, \dots, x_d, y]\!]/(f + y^n)$. If (φ, ψ) is a reduced matrix factorization of f corresponding to the MCM R-module $M = \operatorname{cok}(\varphi, \psi)$, then we have $\Omega_{R^\#}(M) \cong \operatorname{cok}\Omega((\varphi,\psi)\hat{\otimes}Y))$.*

Proof. The proof given in [12, Lemma 8.17] or [19, Lemma 12.3] in the case $n = 2$ applies equally well for arbitrary n. □

Example 5.2. Consider $R = k[\![x]\!]/(x^4)$ and $R^\# = k[\![x, y]\!]/(x^4 + y^3)$, which is a simple curve singularity of type $\mathbb{E}_6$. For details on the matrix factorizations of $x^4 + y^3$ and the category $\mathrm{MCM}(R^\#)$, we refer the reader to (9.13) in [19], and we follow the notation there with the exception that we have swapped x and y in our definition of $R^\#$.

Of course R is a finite-dimensional k-algebra of finite representation type, and R-mod is easily pictured via its Auslander–Reiten quiver

$$\begin{tikzcd} R/(x) \arrow[r, shift left] & R/(x^2) \arrow[l, shift left] \arrow[r, shift left] & R/(x^3) \arrow[l, shift left] \arrow[r, shift left] & R \arrow[l, shift left] \end{tikzcd}.$$

For reference we also provide the Auslander–Reiten quiver of $R^\#$.

$$\begin{tikzcd} & & B \arrow[r] \arrow[dl, shift left] \arrow[dd, dashed, no head] & M_1 \arrow[dl] \arrow[dd, dashed, no head] & \\ M_2 \arrow[r, shift left] \arrow[loop below, dashed, no head] & X \arrow[l, shift left] \arrow[ur, shift left] \arrow[dr, shift left] \arrow[loop below, dashed, no head] & & & R \arrow[ul] \\ & & A \arrow[r] \arrow[ul, shift left] & N_1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We have

$$\begin{aligned} \Omega_{R^\#}(R/(x)) &\cong \operatorname{cok}((x^3, x)\hat{\otimes}(y, y^2)) = \operatorname{cok}\left(\begin{bmatrix} x^3 & -y \\ y^2 & x \end{bmatrix}, \begin{bmatrix} x & y \\ -y^2 & x^3 \end{bmatrix} \right) \\ &\cong \operatorname{cok}(\psi_1, \varphi_1) \cong N_1 ; \end{aligned}$$

and

$$\Omega_{R^\#}(R/(x^3)) \cong \operatorname{cok}((x, x^3)\hat{\otimes}(y, y^2)) \cong \Omega(N_1) \cong M_1 .$$

Furthermore

$$\begin{aligned}\Omega_{R^{\#}}(R/(x^2)) &\cong \operatorname{cok}((x^2,x^2)\hat{\otimes}(y,y^2)) = \operatorname{cok}\left(\begin{bmatrix} x^2 & -y \\ y^2 & x^2\end{bmatrix}, \begin{bmatrix} x^2 & y \\ -y^2 & x^2\end{bmatrix}\right)\\ &\cong \operatorname{cok}(\psi_2,\varphi_2) \cong N_2 \cong M_2\,.\end{aligned}$$

Thus $\Sigma_1 = \operatorname{add}(M_1 \oplus N_1 \oplus M_2 \oplus R^{\#})$. We compute the minimal Σ_1-approximations of the remaining indecomposables in $\operatorname{MCM}(R^{\#})$. From the matrix factorizations for A, B and X given in (9.13) of [19] (with x and y swapped), it is easy to see that $A/yA \cong (R/(x^3))^2 \oplus R/(x^2)$, $B/yB \cong (R/(x))^2 \oplus R/(x^2)$ and $X/yX \cong (R/(x^2))^2 \oplus R/(x) \oplus R/(x^3)$. Applying $\Omega_{R^{\#}}$ now yields the middle terms in the following short exact sequences, which are left and right Σ_1-approximations of the modules on the ends:

$$\begin{gathered}0 \longrightarrow B \longrightarrow M_1^2 \oplus M_2 \longrightarrow A \longrightarrow 0\\ 0 \longrightarrow A \longrightarrow N_1^2 \oplus M_2 \longrightarrow B \longrightarrow 0\\ 0 \longrightarrow X \longrightarrow M_1 \oplus N_1 \oplus M_2^2 \longrightarrow X \longrightarrow 0\end{gathered}$$

In light of Proposition 4.4, there must also be short exact sequences with end terms in Σ_1 and each of B, A and X as direct summands of the middle term. In fact these extensions are evidenced by the following almost split sequences

$$\begin{gathered}0 \longrightarrow M_1 \longrightarrow A \longrightarrow N_1 \longrightarrow 0\\ 0 \longrightarrow N_1 \longrightarrow B \oplus R^{\#} \longrightarrow M_1 \longrightarrow 0\\ 0 \longrightarrow M_2 \longrightarrow X \longrightarrow M_2 \longrightarrow 0.\end{gathered}$$

Set $\widetilde{M} = M_1 \oplus N_1 \oplus M_2 \oplus R^{\#}$. The ring $\Lambda = \operatorname{End}_{R^{\#}}(\widetilde{M})$ is isomorphic to the corner ring of the Auslander algebra of $\operatorname{MCM}(R^{\#})$ corresponding to the 4 vertices M_1, N_1, M_2 and $R^{\#}$ in the AR-quiver above. To describe this algebra explicitly in terms of quivers and relations, we identify $R^{\#}$ with $k[\![t^3,t^4]\!]$, and note that all the indecomposables in Σ_1 can be represented as fractional ideals:

$$M_1 \cong (t^3,t^8), \quad N_1 \cong (t^3,t^4), \quad M_2 \cong (t^6,t^8)\,.$$

The irreducible morphisms between the indecomposables in Σ_1 are easily seen to correspond to the following paths in the AR-quiver of $\operatorname{MCM}(R^{\#})$. Moreover each is realized by multiplication by a suitable power of t.

$$\begin{aligned} t^5 &: N_1 \longrightarrow B \longrightarrow X \longrightarrow M_2,\\ t^4 &: N_1 \longrightarrow B \longrightarrow X \longrightarrow B \longrightarrow M_1,\\ 1 &: N_1 \longrightarrow R^{\#},\\ t^3 &: R^{\#} \longrightarrow M_1,\\ t^3 &: M_1 \longrightarrow A \longrightarrow X \longrightarrow M_2,\end{aligned}$$

$$\begin{aligned}
1 &: M_1 \longrightarrow A \longrightarrow X \longrightarrow A \longrightarrow N_1, \\
t^{-2} &: M_2 \longrightarrow X \longrightarrow A \longrightarrow N_1, \\
1 &: M_2 \longrightarrow X \longrightarrow B \longrightarrow M_1, \\
t^2 &: M_2 \longrightarrow X \longrightarrow A \longrightarrow X \longrightarrow M_2.
\end{aligned}$$

It follows that Λ can be described as a factor of the completed path algebra of the following quiver.[1]

$$\begin{tikzcd}
 & M_2 \arrow[loop above, "t^2"] \arrow[dl, shift left, "1"] \arrow[dr, shift left, "t^{-2}"] & \\
M_1 \arrow[ur, shift left, "t^3"] \arrow[rr, shift left, "1"] & & N_1 \arrow[ll, shift left, "t^4"] \arrow[ul, shift left, "t^5"] \arrow[dl, "1"] \\
 & R^{\#} \arrow[ul, "t^3"] &
\end{tikzcd}$$

While we do not list all the relations here, we note that they can be easily identified from the above quiver, as they correspond to parallel paths that compose to identical powers of t. For instance, among the minimal relations we find the difference between the paths $M_2 \xrightarrow{1} M_1 \xrightarrow{1} N_1$ and $M_2 \xrightarrow{t^2} M_2 \xrightarrow{t^{-2}} N_1$ since both paths compose to 1, as well as the difference between $N_1 \longrightarrow R^{\#} \longrightarrow M_1 \longrightarrow N_1$ and $N_1 \longrightarrow M_2 \longrightarrow N_1$ since both compose to t^3.

Next, we apply results from Section 3 to describe the minimal projective resolutions of the simple Λ-modules. Recall that these will become periodic of period 2 after the first two terms (since $m = \max\{2, d+1\} = 2$ here). If $S(i)$ is a simple left Λ-module, corresponding to a vertex i of the quiver of Λ, its minimal projective presentation is given by

$$\bigoplus_{\alpha: j \longrightarrow i} P(j) \xrightarrow{\pi(i)} P(i) \longrightarrow S(i) \longrightarrow 0,$$

where the sum ranges over all arrows α ending at i, and the α component of $\pi(i)$ is just the map $\alpha\colon P(j) \longrightarrow P(i)$. In general, we can find a map $d(i)$ between $X, Y \in \operatorname{add}(\widetilde{M})$ so that $\pi(i)$ is realized as

$$\operatorname{Hom}_{R^{\#}}(\widetilde{M}, d(i))\colon \operatorname{Hom}_{R^{\#}}(\widetilde{M}, X) \longrightarrow \operatorname{Hom}_{R^{\#}}(\widetilde{M}, Y)$$

[1] With the convention that arrows are composed left to right.

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and $\ker d(i) \in \mathrm{MCM}(R^{\#})$ with $\mathrm{Hom}_{R^{\#}}(\widetilde{M}, \ker d(i)) \cong \Omega^2_{\Lambda}(S(i))$. Furthermore the $\mathrm{add}(\widetilde{M})$-resolution of $\ker d(i)$ will induce the remaining terms of the minimal projective resolution of $S(i)$ over Λ. For example, for $S(N_1)$, the minimal projective presentation has the form

$$P(M_1) \oplus P(M_2) \xrightarrow{\pi(N_1)} P(N_1) \longrightarrow S(N_1) \longrightarrow 0$$

where $\pi(N_1)$ is the map induced by $\begin{pmatrix} 1 \\ t^{-2} \end{pmatrix} : M_1 \oplus M_2 \longrightarrow N_1$, whose kernel is isomorphic to B. Using the Σ_1-approximations of B and A, we now obtain the minimal projective resolution

$$\begin{aligned} \cdots \longrightarrow P(M_1)^2 \oplus P(M_2) \longrightarrow P(N_1)^2 \oplus P(M_2) \longrightarrow P(M_1) \oplus P(M_2) \longrightarrow P(N_1) \\ \longrightarrow S(N_1) \longrightarrow 0 \,. \end{aligned}$$

Similarly for $S(M_2)$, we compute $\Omega^2_{\Lambda}(S(M_2)) \cong \mathrm{Hom}_{R^{\#}}(\widetilde{M}, \ker d(M_2))$, where

$$d(M_2) = \begin{pmatrix} t^3 \\ t^5 \\ t^2 \end{pmatrix} : M_1 \oplus N_1 \oplus M_2 \longrightarrow M_2 \,.$$

One can compute $\ker d(M_2) \cong X$. Then using the Σ_1-approximation sequence for X, we will get the minimal projective resolution

$$\begin{aligned} \cdots \longrightarrow P(M_1) \oplus P(N_1) \oplus P(M_2)^2 \longrightarrow P(M_1) \oplus P(N_1) \oplus P(M_2)^2 \\ \longrightarrow P(M_1) \oplus P(N_1) \oplus P(M_2) \longrightarrow P(M_2) \longrightarrow S(M_2) \longrightarrow 0. \end{aligned}$$

Example 5.3. Similar computations can be made for the $\mathbb{E}_8$ curve singularity $R^{\#} = k[\![x, y]\!]/(x^5 + y^3) \cong k[\![t^3, t^5]\!]$. For details on $\mathrm{MCM}(R^{\#})$ we refer to (9.15) in [19], and we follow the notation introduced there, but again with the exception that x and y are swapped. We set $R = k[x]/(x^5)$, and see that

$$\begin{aligned} \Omega_{R^{\#}}(R/(x)) &\cong \mathrm{cok}((x^4, x) \hat{\otimes} (y, y^2)) = \mathrm{cok}\left(\begin{bmatrix} x^4 & -y \\ y^2 & x \end{bmatrix}, \begin{bmatrix} x & y \\ -y^2 & x^4 \end{bmatrix} \right) \\ &\cong \mathrm{cok}(\psi_1, \varphi_1) \cong N_1 \,; \end{aligned}$$

and

$$\begin{aligned} \Omega_{R^{\#}}(R/(x^2)) &\cong \mathrm{cok}((x^3, x^2) \hat{\otimes} (y, y^2)) = \mathrm{cok}\left(\begin{bmatrix} x^3 & -y \\ y^2 & x^2 \end{bmatrix}, \begin{bmatrix} x^2 & y \\ -y^2 & x^3 \end{bmatrix} \right) \\ &\cong \mathrm{cok}(\psi_2, \varphi_2) \cong N_2; \end{aligned}$$

from which it follows that $\Omega_{R^{\#}}(R/(x^3)) \cong \Omega_{R^{\#}}(N_2) \cong M_2$ and $\Omega_{R^{\#}}(R/(x^4)) \cong \Omega_{R^{\#}}(N_1) \cong M_1$. Thus Σ_1 contains $R^{\#}$ along with the indecomposables $M_1 \cong$

$(t^3, t^{10}), N_1 \cong (t^3, t^5), M_2 \cong (t^6, t^{10})$ and $N_2 \cong (t^5, t^6)$. As before, the irreducible morphisms between these indecomposable $R^{\#}$-modules can all be realized as multiplication by powers of t, and we obtain the following quiver for Λ.

$$\begin{tikzcd}
M_2 \arrow[rr, shift left=2, "{1,t^{-1}}"] \arrow[rr, shift left=0.7] \arrow[d, shift left, "1"] & & N_2 \arrow[ll, shift left=0.7, "{t^4,t^5}"] \arrow[d, shift left, "1"] \\
M_1 \arrow[u, shift left, "t^3"] \arrow[rr, shift left, "1"] & & N_1 \arrow[u, shift left, "t^3"] \arrow[ll, shift left, "t^5"] \arrow[dl, "1"] \\
& R^{\#} \arrow[ul, "t^3"] &
\end{tikzcd}$$

Thus, Λ is isomorphic to a quotient of the completed path algebra of the above quiver, with relations defined by the labels as in the previous example.

We also list the Σ_1-approximation sequences for the remaining indecomposable MCM $R^{\#}$-modules. As in the previous example, for each $M \in \mathrm{MCM}(R^{\#})$ the corresponding sequence is computed by first calculating $M/yM \in \mathrm{MCM}(R)$, which is easily done by looking at the matrix factorization associated to M. We obtain the following sequences (and their syzygies):

$$\begin{aligned}
&0 \longrightarrow B_1 \longrightarrow M_1^2 \oplus N_2 \longrightarrow A_1 \longrightarrow 0 \\
&0 \longrightarrow B_2 \longrightarrow M_1 \oplus M_2^2 \longrightarrow A_2 \longrightarrow 0 \\
&0 \longrightarrow D_i \longrightarrow M_1 \oplus M_2 \oplus N_1 \oplus N_2 \longrightarrow C_i \longrightarrow 0 \quad (i = 1, 2) \\
&0 \longrightarrow Y_1 \longrightarrow M_1 \oplus M_2^2 \oplus N_1 \oplus N_2^2 \longrightarrow X_1 \longrightarrow 0 \\
&0 \longrightarrow Y_2 \longrightarrow M_1^2 \oplus M_2 \oplus N_2^2 \longrightarrow X_2 \longrightarrow 0
\end{aligned}$$

Finally, we mention that the short exact sequences realizing each indecomposable $M \in \mathrm{MCM}(R^{\#})$ as a direct summand of an extension of modules in Σ_1 are not as apparent here as they were in our previous example. Here, only the modules A_1, B_1, C_2 and D_2 arise in the middle terms of almost split sequences ending with objects in Σ_1. We sketch the construction of this sequence in one other example. Consider the module

$$A_2 \cong \operatorname{cok} \begin{bmatrix} x & -y & 0 \\ 0 & x^2 & -y \\ y & 0 & x^2 \end{bmatrix}.$$

As in the proof of Proposition 4.4, we obtain the desired short exact sequence by applying $\Omega_{R^{\#}}$ to the sequence

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$$0 \longrightarrow yA_2/y^2A_2 \longrightarrow A_2/y^2A_2 \longrightarrow A_2/yA_2 \longrightarrow 0.$$

An easy computation shows that both yA_2/y^2A_2 and A_2/yA_2 are isomorphic to $R/(x) \oplus (R/(x^2))^2$ as R-modules. Hence, using Proposition 2.5 we obtain the short exact sequence

$$0 \longrightarrow N_1 \oplus N_2^2 \longrightarrow A_2 \oplus B_2 \oplus F \longrightarrow N_1 \oplus N_2^2 \longrightarrow 0$$

for some free module F. Since A_2, N_1 and N_2 have rank 1, while B_2 has rank 2, we have $F \cong (R^\#)^3$.

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References

[1] Maurice Auslander, Øyvind Solberg, Relative homology and representation theory I. Relative homology and homologically finite subcategories, Comm. Algebra 21 (9) (1993) 2995–3031.

[2] Matthew Ballard, David Favero, Ludmil Katzarkov, Orlov spectra: bounds and gaps, Invent. Math. 189 (2012) 359–430.

[3] Ragnar-Olaf Buchweitz, Maximal Cohen–Macaulay modules and Tate-cohomology over Gorenstein rings, unpublished manuscript, available from http://hdl.handle.net/1807/16682, 1986.

[4] Hailong Dao, Ryo Takahashi, The dimension of a subcategory of modules, Forum Math. Sigma 3 (2015) e19, MR 3482266.

[5] John A. Eagon, D.G. Northcott, Ideals defined by matrices and a certain complex associated with them, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 269 (1962) 188–204, MR 0142592.

[6] Henrik Holm, The category of maximal Cohen–Macaulay modules as a ring with several objects, Mediterr. J. Math. (2015) 1–14 (in English).

[7] Jürgen Herzog, Dorin Popescu, Thom-Sebastiani problems for maximal Cohen–Macaulay modules, Math. Ann. 309 (4) (1997) 677–700, MR 1483829.

[8] Osamu Iyama, Martin Kalck, Michael Wemyss, Dong Yang, Frobenius categories, Gorenstein algebras and rational surface singularities, Compos. Math. 151 (3) (2015) 502–534.

[9] Osamu Iyama, Auslander correspondence, Adv. Math. 210 (1) (2007) 51–82, MR 2298820.

[10] Horst Knörrer, Cohen–Macaulay modules on hypersurface singularities. I, Invent. Math. 88 (1) (1987) 153–164, MR 877010.

[11] Graham J. Leuschke, Endomorphism rings of finite global dimension, Canad. J. Math. 59 (2) (2007) 332–342, MR 2310620.

[12] Graham J. Leuschke, Roger Wiegand, Cohen–Macaulay Representations, Mathematical Surveys and Monographs, vol. 181, American Mathematical Society, Providence, RI, ISBN 978-0-8218-7581-0, 2012, xviii+367 pp., MR 2919145.

[13] Takehiko Miyata, Note on direct summands of modules, J. Math. Kyoto Univ. 7 (1967) 65–69, MR 0214585.

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[14] Liam O'Carroll, Dorin Popescu, On a theorem of Knörrer concerning Cohen–Macaulay modules, in: Commutative Algebra, Homological Algebra and Representation Theory, Catania/Genoa/Rome, 1998, J. Pure Appl. Algebra 152 (1–3) (2000) 293–302, MR 1784003 (2001g:13020).

[15] Dmitri O. Orlov, Derived categories of coherent sheaves and triangulated categories of singularities, in: Algebra, Arithmetic, and Geometry: in Honor of Yu.I. Manin, vol. II, in: Progr. Math., vol. 270, Birkhäuser Boston Inc., Boston, MA, 2009, pp. 503–531, MR 2641200.

[16] Christopher L. Quarles, Krull–Schmidt Rings & Noncommutative Resolutions of Singularities, Master's thesis, University of Washington, 2005.

[17] Raphaël Rouquier, Dimensions of triangulated categories, J. K-Theory 1 (2) (2008) 193–256, MR 2434186.

[18] Ryo Takahashi, Classifying thick subcategories of the stable category of Cohen–Macaulay modules, Adv. Math. 225 (4) (2010) 2076–2116, MR 2680200.

[19] Yuji Yoshino, Cohen–Macaulay Modules Over Cohen–Macaulay Rings, London Mathematical Society Lecture Note Series, vol. 146, Cambridge University Press, Cambridge, 1990, MR 1079937.

[20] Yuji Yoshino, Tensor products of matrix factorizations, Nagoya Math. J. 152 (1998) 39–56.

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