Compositio Math. (Page 1 of 23)
doi:10.1112/S0010437X14008070

On the derived category of Grassmannians in arbitrary characteristic

Ragnar-Olaf Buchweitz, Graham J. Leuschke and Michel Van den Bergh

Abstract

In this paper we consider Grassmannians in arbitrary characteristic. Generalizing Kapranov's well-known characteristic-zero results, we construct dual exceptional collections on them (which are, however, not strong) as well as a tilting bundle. We show that this tilting bundle has a quasi-hereditary endomorphism ring and we identify the standard, costandard, projective and simple modules of the latter.

1. Introduction

Throughout K is a field of arbitrary characteristic. Let X be a smooth algebraic variety over K and let $\mathcal{D}$ be its bounded derived category of coherent sheaves. An object $\mathcal{T} \in \mathcal{D}$ is called a *tilting object* if it classically generates $\mathcal{D}$ (i.e. the smallest thick subcategory of $\mathcal{D}$ containing $\mathcal{T}$ is $\mathcal{D}$ itself) and $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{T}, \mathcal{T}[i]) = 0$ for $i \neq 0$.

If $\mathcal{T}$ is a tilting object in $\mathcal{D}$ and $A = \operatorname{End}_{\mathcal{O}_X}(\mathcal{T})$ then the functor $\operatorname{RHom}_{\mathcal{O}_X}(\mathcal{T}, -)$ defines an equivalence $\mathcal{D} \cong D^b(\operatorname{mod} A^\circ)$. If, in addition, $\mathcal{T}$ is a vector bundle then we call $\mathcal{T}$ a *tilting bundle*.

A sequence of objects $E_1, E_2, \dots, E_d$ which classically generates $\mathcal{D}$ is called an *exceptional sequence* if $\operatorname{RHom}_{\mathcal{O}_X}(E_j, E_i) = 0$ for $j > i$ and $\operatorname{RHom}_{\mathcal{O}_X}(E_i, E_i) = K$. An exceptional sequence is *strongly exceptional* if, in addition, $\operatorname{Ext}^k_{\mathcal{O}_X}(E_i, E_j) = 0$ for all i, j and $k \neq 0$. Obviously if $(E_i)_i$ is strongly exceptional then $\mathcal{T} = \bigoplus_i E_i$ is a tilting object in $\mathcal{D}$.

Two exceptional sequences $E_1, E_2, \dots, E_d$ and $F_d, F_{d-1}, \dots, F_1$ are said to be *dual* if

$$\operatorname{RHom}_{\mathcal{O}_X}(E_i, F_j) = \delta_{i,j} \cdot K.$$

We now specialize to the case where X is the Grassmannian $\mathbb{G} = \operatorname{Grass}(l, F) \cong \operatorname{Grass}(l, m)$ of l-dimensional subspaces of an m-dimensional K-vector space F. On $\mathbb{G}$ we have a tautological exact sequence of vector bundles

$$0 \longrightarrow \mathcal{R} \longrightarrow F^\vee \otimes_K \mathcal{O}_{\mathbb{G}} \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{1.1}$$

in which $\mathcal{Q}$ has rank l and $\mathcal{R}$ has rank $m - l$. When K is a field of characteristic zero, Kapranov [Kap88] constructs a pair of dual strongly exceptional sequences on $\mathbb{G}$ which we now

Received 11 April 2013, accepted in final form 2 October 2014.
2010 Mathematics Subject Classification 14F05, 14M15 (primary), 16S38, 15A75 (secondary).
Keywords: Grassmannian variety, exceptional collection, tilting bundle, semi-orthogonal decomposition, quasi-hereditary algebra.
The first author was partly supported by NSERC grant 3-642-114-80. The second author was partly supported by NSF grant DMS 0902119 and a grant from the NSA. The third author is director of research at the FWO. Part of this research was supported through the 'Research in Pairs' programme by the Mathematisches Forschungsinstitut Oberwolfach in July 2012, and the work was completed at MSRI. We thank both institutions for their hospitality.

describe. For a partition α, let L^α be the associated Schur functor; our conventions are that $L^{(t)}V = \operatorname{Sym}^t V$ and $L^{(1^t)}V = \bigwedge^t V$. Further, let α' be the transpose of α and let $|\alpha| = \sum_i \alpha_i$ be its degree.

Theorem 1.1 (Kapranov [Kap88]). *Assume that K has characteristic zero. Let $B_{u,v}$ be the set of partitions with at most u rows and at most v columns equipped with a total ordering $\prec$ such that if $|\alpha| < |\beta|$ then $\alpha \prec \beta$. Let $\overline{B}_{u,v}$ be the same as $B_{u,v}$ but equipped with the opposite ordering. Then there are strongly exceptional sequences on $\mathbb{G}$ given by*

$$(L^\alpha \mathcal{Q})_{\alpha\in B_{l,m-l}} \quad \text{and} \quad (L^{\alpha'}\mathcal{R})_{\alpha\in \overline{B}_{l,m-l}}.$$

In particular, the vector bundle

$$\mathcal{K} = \bigoplus_{\alpha\in B_{l,m-l}} L^\alpha \mathcal{Q}$$

is a tilting bundle on $\mathbb{G}$. Moreover, the exceptional sequences $(L^\alpha \mathcal{Q})_{\alpha\in B_{l,m-l}}$ and $(L^{\alpha'}\mathcal{R}[|\alpha|])_{\alpha\in \overline{B}_{l,m-l}}$ are dual.

We remark that there are many total orderings $\prec$ refining the partial ordering on partitions given by degree.

For K a field of positive characteristic p, Kaneda [Kan08] shows that $\mathcal{K}$ remains tilting as long as $p \geqslant m-1$. However, $\mathcal{K}$ fails to be tilting in very small characteristics.

Example 1.2. Assume that K has characteristic two and put $\mathbb{G} = \operatorname{Grass}(2,4)$. Then the short exact sequence

$$0 \longrightarrow \textstyle\bigwedge^2 \mathcal{Q} \longrightarrow \mathcal{Q}\otimes\mathcal{Q} \longrightarrow \operatorname{Sym}^2\mathcal{Q} \longrightarrow 0 \tag{1.2}$$

is non-split. This follows, for example, from Theorem 5.4 below and the fact that the sequence of GL(2)-representations

$$0 \longrightarrow \textstyle\bigwedge^2 V \longrightarrow V\otimes V \longrightarrow \operatorname{Sym}^2 V \longrightarrow 0$$

is not split, where $V = K^2$ is the standard representation. In particular, $\operatorname{Ext}^1_{\mathcal{O}_\mathbb{G}}(\operatorname{Sym}^2\mathcal{Q}, \bigwedge^2\mathcal{Q}) \neq 0$, so that $\operatorname{Sym}^2\mathcal{Q}$ and $\bigwedge^2\mathcal{Q}$ are not common direct summands of a tilting bundle on $\mathbb{G}$ in characteristic two.

In this note we give a tilting bundle on $\mathbb{G}$ which exists in arbitrary characteristic. For a partition $\alpha = [\alpha_1,\ldots,\alpha_p]$, put

$$\textstyle\bigwedge^\alpha \mathcal{Q} = \bigwedge^{\alpha_1}\mathcal{Q}\otimes_\mathbb{G}\cdots\otimes_\mathbb{G}\bigwedge^{\alpha_p}\mathcal{Q}.$$

Our first main theorem is the following.

Theorem 1.3. *Define a vector bundle on $\mathbb{G}$ by*

$$\mathcal{T} = \bigoplus_{\alpha\in B_{l,m-l}} \textstyle\bigwedge^{\alpha'}\mathcal{Q}. \tag{1.3}$$

Then $\mathcal{T}$ is a tilting bundle on $\mathbb{G}$.

In characteristic zero we recover Kapranov's tilting bundle, up to multiplicities, by working out the tensor products in (1.3) using Pieri's formula.

The proof of Theorem 1.3 depends on the following vanishing result which we also use in [BLVdB13]. For brevity we write u^v for the partition $(u,u,\ldots,u)$ with v parts.

Proposition 1.4. *For $\alpha \in B_{l,m-l}$ and β an arbitrary partition, we have*

$$\operatorname{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(\textstyle\bigwedge^{\alpha'} \mathcal{Q}, L^\beta \mathcal{Q}) = 0 \tag{1.4}$$

if either $i > 0$ or $|\beta| < |\alpha|$. Furthermore, if $|\beta| = |\alpha|$ then $\dim_K \operatorname{Hom}_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'} \mathcal{Q}, L^\beta \mathcal{Q})$ is equal to the number of semi-standard Young tableaux of shape β' and content α'.

In our next result we show that Kapranov's characteristic-zero result can be partially salvaged in arbitrary characteristic.

Theorem 1.5 (See Theorem 7.5 below). *There exists a total ordering $\prec$ on $B_{l,m-l}$ such that*

$$(L^\alpha \mathcal{Q})_{\alpha \in B_{l,m-l}} \quad \text{and} \quad (L^{\alpha'} \mathcal{R}[|\alpha|])_{\alpha \in \overline{B}_{l,m-l}}$$

are dual exceptional collections on $\mathbb{G}$, where $\overline{B}_{l,m-l}$ is $B_{l,m-l}$ equipped with the opposite ordering.

We use this result to obtain another proof of Kaneda's result that $\mathcal{K}$ remains tilting in characteristics $p \geqslant m-1$ (Corollary 7.8).

The proof of Theorem 1.5 goes through the construction of a nice semi-orthogonal decomposition [BK89] on $D^b(\operatorname{coh}(\mathbb{G}))$ which we summarize in the following theorem.

Theorem 1.6 (See Theorem 5.6 below). *There is a semi-orthogonal decomposition*

$$D^b(\operatorname{coh}(\mathbb{G})) = \langle \mathcal{D}_0, \dots, \mathcal{D}_{l(m-l)} \rangle$$

where for $d = 0, \dots, l(m-l)$, $\mathcal{D}_d$ is the derived category of the generalized Schur algebra associated to the $\mathrm{GL}(l)$-representations whose composition factors have highest weight $\alpha \in B_{l,m-l}$ such that $|\alpha| = d$.

The connection between Theorems 1.3 and 1.5 depends on the theory of quasi-hereditary algebras [DR92]. In this regard we have the following additional result.

Theorem 1.7. *Let $\mathcal{T}$ be as in Theorem 1.3 and put $A = \operatorname{End}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T})$. Then A is quasi-hereditary. Furthermore, the homogeneous bundles $(L^\alpha \mathcal{Q})_{\alpha \in B_{l,m-l}}$ correspond to the standard right A-modules and $(L^{\alpha'} \mathcal{R}[|\alpha|])_{\alpha \in \overline{B}_{l,m-l}}$ correspond to the costandard right A-modules.*

This theorem is a special case of Theorem 8.3 below in which we also characterize the simple and projective right A-modules.

2. Some preliminaries on representation theory

Throughout we use [Jan03] as a convenient reference for facts about algebraic groups. If $H \subset G$ is an inclusion of algebraic groups over the ground field K, then the restriction functor from rational G-modules to rational H-modules has a right adjoint denoted by ind_H^G [Jan03, I.3.3]. Its right derived functors are denoted by $R^i \operatorname{ind}_H^G$. For an inclusion of groups $K \subset H \subset G$ and M a rational K-representation, there is a spectral sequence [Jan03, I.4.5(c)]

$$E_2^{pq} : R^p \operatorname{ind}_H^G R^q \operatorname{ind}_K^H M \Longrightarrow R^{p+q} \operatorname{ind}_K^G M. \tag{2.1}$$

If H is a closed subgroup scheme of G, so that G/H is a scheme, and V is a finite-dimensional H-representation, then $\mathcal{L}_{G/H}(V)$ is by definition the G-equivariant vector bundle on G/H given by the sections of $(G \times V)/H$. The functor $\mathcal{L}_{G/H}(-)$ defines an equivalence [CPS83, Theorem 2.7] between the finite-dimensional H-representations and the G-equivariant vector bundles on

G/H. The inverse of this functor is given by taking the fiber in $[H] \in G/H$. Furthermore, in this situation the right derived functors $R^i \operatorname{ind}_H^G$ may be computed as [Jan03, Proposition I.5.12]

$$R^i \operatorname{ind}_H^G M = H^i(G/H, \mathcal{L}_{G/H}(M)). \tag{2.2}$$

We now assume that G is a split reductive group with a given split maximal torus and Borel $T \subset B \subset G$. We let $X(T)$ be the character group of T and we identify the elements of $X(T)$ with the one-dimensional representations of T. We also consider characters of T as characters of B via the canonical projection $B \longrightarrow T$. The set of roots (the weights of G on $\operatorname{Lie} G$) is denoted by R. We have $R = R^- \coprod R^+$, where the negative roots R^- represent the roots of $\operatorname{Lie} B$. For $\alpha \in R$, we denote the corresponding coroot in $Y(T) = \operatorname{Hom}(X(T), \mathbb{Z})$ [Jan03, II.1.3] by $\alpha^\vee$. The natural pairing between $X(T)$ and $Y(T)$ is denoted by $\langle -, - \rangle$. A weight $\lambda \in X(T)$ is dominant if $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for all positive roots α. The set of dominant weights is denoted by $X(T)_+$. The set $X(T)$ is naturally partially ordered by putting $\lambda \leqslant \mu$ if $\mu - \lambda$ is a sum of positive roots.

The following is the celebrated Kempf vanishing result ([Kem76]; see also [Jan03, II.4.5]).

Theorem 2.1. *If $\lambda \in X(T)_+$ then $R^i \operatorname{ind}_B^G \lambda = H^i(G/B, \mathcal{L}_{G/B}(\lambda))$ vanishes for all strictly positive i.*

We now restrict to $G = \operatorname{GL}(m)$. In this case we let T be the diagonal matrices in G, and B the lower triangular matrices. The weights of T can be identified with m-tuples of integers $[\alpha_1, \dots, \alpha_m]$ via $\operatorname{diag}(z_1, \dots, z_m) \mapsto z_1^{\alpha_1} \cdots z_m^{\alpha_m}$. Thus $X(T) \cong Y(T) \cong \mathbb{Z}^m$. Under this identification roots and coroots coincide and are given by $(0, \dots, 0, \pm 1, 0, \dots, 0, \mp 1, 0, \dots, 0)$. The pairing between $X(T)$ and $Y(T)$ is the standard Euclidean scalar product and hence $X(T)_+ = \{[\alpha_1, \dots, \alpha_m] \mid \alpha_i \geqslant \alpha_j \text{ for } i \leqslant j\}$. A dominant weight with only non-negative entries will be called a *partition*. Mentally we represent a partition by its Young diagram, with the length of the rows corresponding to the entries. The sum $\sum_i \alpha_i$ is the *degree* of the weight α and is denoted by $|\alpha|$. We say that a representation has degree d if all its weights have degree d. We say that a representation is *polynomial* if all its weights contain only non-negative entries.

If $\alpha = [\alpha_1, \dots, \alpha_m]$ is a partition then we denote by L^α, K^α the corresponding *Schur* and *Weyl functors*. More precisely, for a vector space (or a vector bundle) V, define, for a partition α,

$$\textstyle\bigwedge^\alpha V = \bigotimes_i \bigwedge^{\alpha_i} V, \qquad \operatorname{Sym}^\alpha V = \bigotimes_i \operatorname{Sym}^{\alpha_i} V, \qquad D^\alpha V = \bigotimes_i D^{\alpha_i} V,$$

where in particular $D^u V = (V^{\otimes u})^{S_u}$ is the uth divided power representation, $\bigwedge^u V$ is the exterior power, and $\operatorname{Sym}^u V$ is the symmetric power. The exterior power $\bigwedge^u V$ is self-dual, while the divided power and symmetric power are dual to each other, $(D^u V)^\vee \cong \operatorname{Sym}^u(V^\vee)$.

Then we put, with $d = |\alpha|$,

$$L^\alpha V = \operatorname{im}(\textstyle\bigwedge^{\alpha'} V \xrightarrow{a^\vee} V^{\otimes d} \xrightarrow{s} \operatorname{Sym}^\alpha V), \tag{2.3}$$

$$K^\alpha V = \operatorname{im}(D^\alpha V \xrightarrow{s^\vee} V^{\otimes d} \xrightarrow{a} \textstyle\bigwedge^{\alpha'} V), \tag{2.4}$$

where a and s are respectively the anti-symmetrization map and the symmetrization map. Their precise form is obtained from a filling of the Young diagram associated to α (see [Ful97, § 8.1]). The resulting representations $K^\alpha V$, $L^\alpha V$ are independent of this labeling. Furthermore, we have $(K^\alpha V)^\vee \cong L^\alpha(V^\vee)$.

In the sequel we freely pass between the functor point of view and the representation theory point of view using the following lemma. If $\lambda \in X(T)_+$ then $H^0(\lambda) \overset{\text{def}}{=} \operatorname{ind}_B^G \lambda$ is a so-called *induced representation* with highest weight λ. Dually one defines the corresponding *Weyl representation* as $V(\lambda) = H^0(-w_0\lambda)^\vee$ where w_0 is the longest element of the Weyl group [Jan03, § 2.13].

Lemma 2.2. *Let V be the standard representation of $\mathrm{GL}(m)$ and let α be a partition. Then*

$$L^\alpha V = H^0(\alpha), \tag{2.5}$$
$$K^\alpha V = V(\alpha) \tag{2.6}$$

Proof. The identity (2.5) is [Wey03, (4.3.1)].[1] To prove (2.6) we note that by [Jan03, II.2.13(2)] we have $V(\alpha) = {}^\tau H^0(\alpha)$, where ${}^\tau M$ is $M^\vee$ as a vector space and $g \in G$ acts on $\varphi \in M^\vee$ via $g \cdot \varphi = \varphi \circ g^t$ where $(-)^t$ denotes transposition. Clearly $M \mapsto {}^\tau M$ is a contravariant monoidal functor and, furthermore, one verifies

$$ {}^\tau \mathrm{Sym}^u V = D^u V,$$
$$ {}^\tau {\textstyle\bigwedge}^u V = {\textstyle\bigwedge}^u V.$$

Applying ${}^\tau(-)$ to the right-hand side of (2.3) yields the right-hand side of (2.4), finishing the proof. □

According to [Jan03, Proposition II.2.3], $L^\alpha V$ has a simple socle which we denote by Σ^α. According to [Jan03, § II.2.14] $K^\alpha V$ has a simple top, which is also equal to Σ^α. (Recall that the socle is the sum of all simple submodules, while the top of a module M is $M/\operatorname{rad} M$.)

We also state for easy reference the following characteristic-free versions of the Cauchy formula and the Littlewood–Richardson rule; see [Wey03, (2.3.2), (2.3.4)].

Theorem 2.3 (Boffi [Bof88], Doubilet *et al.* [DRS74]). *Let V and W be K-vector spaces and let α and β be dominant weights.*

(i) *There is a natural filtration on $\mathrm{Sym}^t(V \otimes W)$ whose associated graded object is a direct sum with summands tensor products $L^\gamma V \otimes L^\gamma W$ of Schur functors with $|\gamma| = t$.*

(ii) *There is a natural filtration on $L^\alpha V \otimes L^\beta V$ whose associated graded object is a direct sum of Schur functors $L^\gamma V$ with $|\gamma| = |\alpha| + |\beta|$. The γ that appear, and their multiplicities, can be computed using the usual Littlewood–Richardson rule: $L^\gamma V$ appears with multiplicity equal to the number of Littlewood–Richardson tableaux of shape γ/α and content β.*

If $\operatorname{char} K = 0$ then the filtrations above degenerate to direct sums.

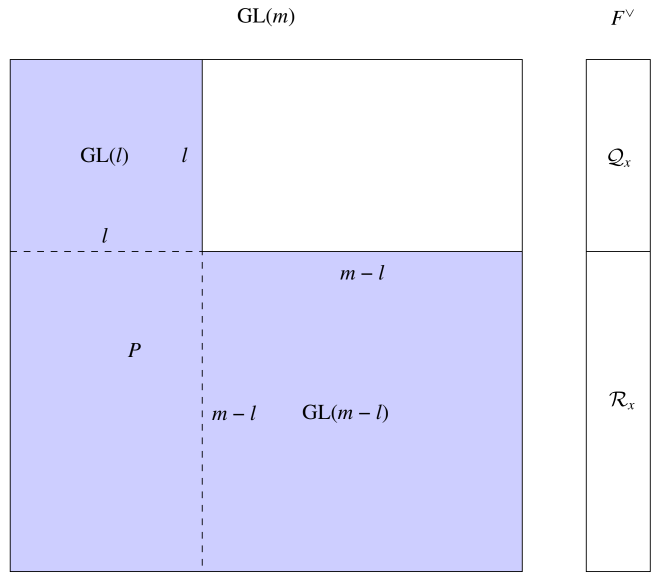
We single out the following consequence of iterating (ii): the multiplicity of a Schur functor $L^\gamma V$ in a tensor product of the form $\bigwedge^\alpha V \otimes L^\beta V$ is equal to the number of semi-standard (i.e. weakly row-increasing and strictly column-increasing) Young tableaux of shape equal to the skew partition γ'/β' and content equal to α, that is, α_1 ones, α_2 twos, etc.

3. Proofs of Theorem 1.3 and Proposition 1.4

We stick to the notation already introduced in the Introduction. We will identify $\mathbb{G} = \mathrm{Grass}(l, F)$ with $\mathrm{Grass}(m-l, F^\vee)$ via the correspondence $(V \subset F) \mapsto ((F/V)^\vee \subset F^\vee)$.

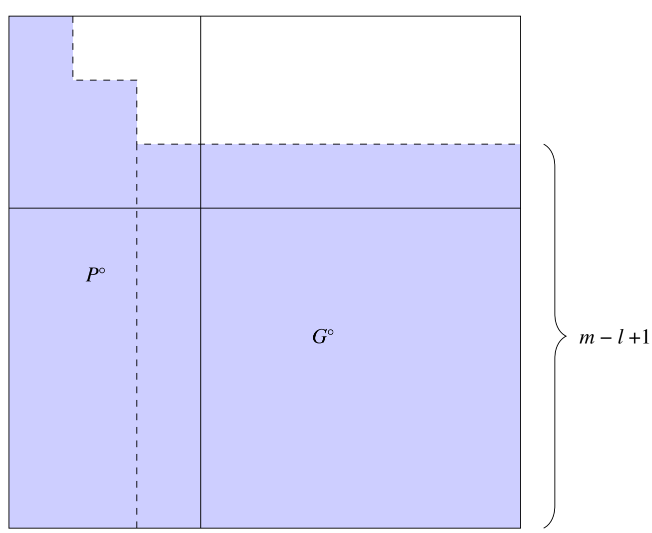
For convenience we choose a basis $(f_i)_{i=1,\ldots,m}$ for F and a corresponding dual basis $(f_i^*)_i$ for $F^\vee$. We view $\mathbb{G}$ as the homogeneous space G/P with $G = \mathrm{GL}(F^\vee) = \mathrm{GL}(m)$ and $P \subset G$ the parabolic subgroup stabilizing the point $(W \subset F^\vee) \in \mathbb{G}$ where $W = \sum_{i=l+1}^m K f_i^*$. As above, let T and B be respectively the diagonal matrices and the lower triangular matrices in G.

[1] Note that our L^α is $L_{\alpha'}$ in [Wey03].



Let $H = G_1 \times G_2 = \mathrm{GL}(l) \times \mathrm{GL}(m-l) \subset \mathrm{GL}(m)$ be the Levi subgroup of P containing T. We put $B_i = B \cap G_i$ and $T_i = T \cap G_i$. We denote the standard representations of G_1 and G_2 by V and W, respectively. Thus, for $x = [P] \in G/P$, we have $V = \mathcal{Q}_x$ and $W = \mathcal{R}_x$, equivalently $\mathcal{Q} = \mathcal{L}_{\mathbb{G}}(V)$ and $\mathcal{R} = \mathcal{L}_{\mathbb{G}}(W)$. (Throughout we silently view G_i-representations as P-representations to apply $\mathcal{L}_{\mathbb{G}}(-)$.) It follows that $\bigwedge^{\alpha'}\mathcal{Q} = \mathcal{L}_{\mathbb{G}}(\bigwedge^{\alpha'} V)$ and $L^\alpha \mathcal{Q} = \mathcal{L}_{\mathbb{G}}(L^\alpha V)$.

For use in the proof below we fix an additional parabolic P° in G given by the stabilizer of the flag $(\sum_{i\geqslant p} Kf_i^*)_{p=1,\dots,l-1}$. We let $G^\circ = \mathrm{GL}(m-l+1) \subset P^\circ \subset G = \mathrm{GL}(m)$ be the lower right $(m-l+1 \times m-l+1)$-block in $\mathrm{GL}(m)$. We put $T^\circ = T \cap G^\circ$ and $B^\circ = B \cap G^\circ$, i.e. B° is the set of lower triangular matrices in G° and T° is the set of diagonal matrices.



Proof of Proposition 1.4. The usual spectral sequence argument implies that $\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'}\mathcal{Q}, L^\beta\mathcal{Q})$ is the ith cohomology of $\mathcal{H}om_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'}\mathcal{Q}, L^\beta\mathcal{Q}) \cong (\bigwedge^{\alpha'}\mathcal{Q})^\vee \otimes_{\mathbb{G}} L^\beta\mathcal{Q}$. If, therefore, $\alpha' = [u_1, \dots, u_{m-l}]$, then we must show

$$H^i(\mathbb{G}, \textstyle\bigwedge^{u_1}\mathcal{Q}^\vee \otimes_{\mathbb{G}} \cdots \otimes_{\mathbb{G}} \bigwedge^{u_{m-l}}\mathcal{Q}^\vee \otimes_{\mathbb{G}} L^\beta\mathcal{Q}) = 0$$

for all $i > 0$, and also for $i = 0$ if $\sum u_i > |\beta|$.

We have, for arbitrary u,

$$(\bigwedge^u \mathcal{Q})^\vee = \bigwedge^{l-u} \mathcal{Q} \otimes_{\mathbb{G}} (\bigwedge^l \mathcal{Q})^\vee, \tag{3.1}$$

and from the tautological exact sequence (1.1),

$$(\bigwedge^l \mathcal{Q})^\vee = \bigwedge^m F \otimes_K \bigwedge^{m-l} \mathcal{R},$$

so (after discarding the one-dimensional vector spaces $\bigwedge^m F$) we must compute the cohomology of

$$\bigwedge^{l-u_1} \mathcal{Q} \otimes_{\mathbb{G}} \cdots \otimes_{\mathbb{G}} \bigwedge^{l-u_{m-l}} \mathcal{Q} \otimes_{\mathbb{G}} L^\beta \mathcal{Q} \otimes_{\mathbb{G}} \bigwedge^{m-l} \mathcal{R} \otimes_{\mathbb{G}} \cdots \otimes_{\mathbb{G}} \bigwedge^{m-l} \mathcal{R}, \tag{3.2}$$

with $m-l$ copies of $\bigwedge^{m-l} \mathcal{R}$.

Now by the Littlewood–Richardson rule (Theorem 2.3(ii)) the tensor factor involving $\mathcal{Q}$ is filtered by subquotients of the form $L^\gamma \mathcal{Q}$ with

$$|\gamma| = |\beta| + (l-u_1) + \cdots (l-u_{m-l}) = |\beta| - |\alpha| + l(m-l). \tag{3.3}$$

We are therefore reduced to computing the cohomology of

$$L^\gamma \mathcal{Q} \otimes_{\mathbb{G}} \bigwedge^{m-l} \mathcal{R} \otimes_{\mathbb{G}} \cdots \otimes_{\mathbb{G}} \bigwedge^{m-l} \mathcal{R}$$

(with $m-l$ factors of $\bigwedge^{m-l} \mathcal{R}$), where γ is an arbitrary partition of degree $|\beta| - |\alpha| + l(m-l)$. Observe for later use that the number of successive subquotients appearing in (3.2) can be computed using the usual Littlewood–Richardson rule as in characteristic zero.

Using (2.2), we see that we must compute

$$R^i \operatorname{ind}_P^G (L^\gamma V \otimes \bigwedge^{m-l} W \otimes \cdots \otimes \bigwedge^{m-l} W), \tag{3.4}$$

where, as above, V, W are the standard representations of G_1, G_2, for all i. Since V has rank l, we may assume that γ has at most l entries. Put $\overline{\gamma} = [\gamma_1, \ldots, \gamma_l, m-l, \ldots, m-l] \in X(T)$. We have

$$\begin{aligned} L^\gamma V \otimes \bigwedge^{m-l} W \otimes \cdots \otimes \bigwedge^{m-l} W &= \operatorname{ind}_{B_1}^{G_1} L^\gamma V \otimes \operatorname{ind}_{B_2}^{G_2} L^{(m-l)^{m-l}} W \\ &= \operatorname{ind}_B^P \overline{\gamma}. \end{aligned}$$

It is clear that $\overline{\gamma}$ is dominant when viewed as a weight for T considered as a maximal torus in $H = G_1 \times G_2$ with respect to the Borel subgroup $B_1 \times B_2$. So Kempf vanishing implies that $R^i \operatorname{ind}_B^P \overline{\gamma} = R^i \operatorname{ind}_{B_1 \times B_2}^{G_1 \times G_2} \overline{\gamma} = 0$ for all $i > 0$.

Thus the spectral sequence (2.1) degenerates and we obtain

$$R^i \operatorname{ind}_P^G (L^\gamma V \otimes \bigwedge^{m-l} W \otimes \cdots \otimes \bigwedge^{m-l} W) = R^i \operatorname{ind}_B^G \overline{\gamma}. \tag{3.5}$$

If $\overline{\gamma}$ is dominant (i.e. $\gamma_l \geqslant m-l$) then the desired vanishing of (3.4) for $i > 0$ follows by invoking Kempf vanishing again.

To finish the case $i > 0$, we therefore assume that $\overline{\gamma}$ is not dominant, i.e. $0 \leqslant \gamma_l < m-l$. We claim that $R^i \operatorname{ind}_B^{P^\circ} \overline{\gamma} = 0$ for all i. Then by the spectral sequence (2.1) applied to $B \subset P^\circ \subset G$ we obtain that $R^i \operatorname{ind}_B^G \overline{\gamma} = 0$ for all i.

To prove the claim we note that $P^\circ / B \cong G^\circ / B^\circ$ and hence, by (2.2), $R^i \operatorname{ind}_B^{P^\circ} \overline{\gamma} = R^i \operatorname{ind}_{B^\circ}^{G^\circ} (\overline{\gamma} \mid T^\circ)$. In other words, we have reduced ourselves to the case $l = 1$ (replacing m by $m - l + 1$).

So now we assume $l = 1$. Thus $\mathbb{G} = \mathbb{P}(F) \cong \mathbb{P}^{m-1}$, which we write as $\mathbb{P}$ for short. The partition γ consists of a single entry γ_1 and we have $\overline{\gamma} = [\gamma_1, m-1, \ldots, m-1]$. Under the assumption

$\gamma_1 < m-1$ we have to prove $R^i \operatorname{ind}_B^G \overline{\gamma} = 0$ for all i. Applying (3.5) in reverse, this means we have to prove that, for $\gamma_1 \geqslant 0$,

$$\mathcal{Q}^{\otimes \gamma_1} \otimes_{\mathbb{P}} (\textstyle\bigwedge^{m-1}\mathcal{R})^{\otimes m-1}$$

has vanishing cohomology on $\mathbb{P}$.

We now observe $\mathcal{Q} \cong \mathcal{O}_{\mathbb{P}}(1)$, and since

$$\mathcal{R} \cong \ker(\mathcal{O}_{\mathbb{P}}^m \longrightarrow \mathcal{O}_{\mathbb{P}}(1)),$$

we also have

$$\textstyle\bigwedge^{m-1}\mathcal{R} \cong \mathcal{O}_{\mathbb{P}}(-1)$$

so that

$$\mathcal{Q}^{\otimes \gamma_1} \otimes_{\mathbb{P}} \textstyle\bigwedge^{m-1}\mathcal{R}^{\otimes m-1} \cong \mathcal{O}_{\mathbb{P}}(-m+1+\gamma_1).$$

It is standard that this line bundle has vanishing cohomology when $0 \leqslant \gamma_1 < m-1$, so we are done with the case $i > 0$.

For the assertions in the case $i = 0$, first note that if $|\beta| < |\alpha|$ then $|\gamma| < l(m-l)$, hence $\gamma_l < m-l$. This means that in the above argument we are always in the case where $\overline{\gamma}$ is not dominant, and thus the vanishing holds also when $i = 0$.

Finally, if $|\beta| = |\alpha|$ then we have $|\gamma| = l(m-l)$. If $\gamma_l < m-l$ then once again $\overline{\gamma}$ is not dominant, and we finish as above. If, on the other hand, $\gamma_l \geqslant m-l$ then we must have $\gamma = [m-l, m-l, \dots, m-l]$ (with l entries). So we must compute the K-dimension of

$$\operatorname{ind}_B^G \overline{\gamma} = H^0(G/B, \mathcal{L}_{G/B}(\overline{\gamma})) \tag{3.6}$$

where $\overline{\gamma} = [m-l, \dots, m-l]$ (with m entries). This is a product of copies of the determinant representation, so is well known to have dimension one.

The desired K-dimension of $\operatorname{Hom}_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'}\mathcal{Q}, L^\beta\mathcal{Q})$ is thus equal to the multiplicity of $L^{(m-l)^l}\mathcal{Q} = (\bigwedge^l \mathcal{Q})^{\otimes(m-l)}$ in the tensor product $\bigwedge^{l^{m-l}-\alpha'}\mathcal{Q} \otimes L^\beta \mathcal{Q}$. By Theorem 2.3, we may compute this multiplicity in characteristic zero, where it is equal to the vector space dimension of

$$\begin{aligned}
&\operatorname{Hom}_G((\textstyle\bigwedge^l V)^{\otimes(m-l)}, \textstyle\bigwedge^{l-u_1}V \otimes \cdots \otimes \textstyle\bigwedge^{l-u_{m-l}}V \otimes L^\beta V)\\
&\quad = [(\textstyle\bigwedge^l V)^{\otimes -(m-l)} \otimes \textstyle\bigwedge^{l-u_1}V \otimes \cdots \otimes \textstyle\bigwedge^{l-u_{m-l}}V \otimes L^\beta V]^G\\
&\quad = [\textstyle\bigwedge^{u_1}V^\vee \otimes \cdots \otimes \textstyle\bigwedge^{u_{m-l}}V^\vee \otimes L^\beta V]^G\\
&\quad = \operatorname{Hom}_G(L^\beta V^\vee,\ \textstyle\bigwedge^{u_1}V^\vee \otimes \cdots \otimes \textstyle\bigwedge^{u_{m-l}}V^\vee)
\end{aligned}$$

using (3.1). This last quantity is equal to the multiplicity of $L^\beta V^\vee$ in $\bigwedge^{u_1}V^\vee \otimes_{\mathbb{G}} \cdots \otimes_{\mathbb{G}} \bigwedge^{u_{m-l}} V^\vee$, which by the remark following Theorem 2.3 is equal to the number of semi-standard tableaux of shape β' and content α'. $\square$

Proof of Theorem 1.3. The main thing to prove is that $\operatorname{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, \mathcal{T}) = 0$ for $i \neq 0$. Applying the characteristic-free Littlewood–Richardson rule Theorem 2.3(ii), we see that it suffices to prove that $\mathcal{T}^\vee \otimes_{\mathbb{G}} L^\gamma \mathcal{Q}$ has vanishing higher cohomology whenever γ is a partition with at most l rows. This follows from Proposition 1.4.

Kapranov's resolution of the diagonal argument together with the characteristic-free version of Cauchy's formula (Theorem 2.3(i)) still implies that the vector bundle $\mathcal{K}$ in Theorem 1.1 classically generates $D^b(\operatorname{coh}(\mathbb{G}))$. See, for example, [LSW89]. Thus it suffices to show that $L^\alpha \mathcal{Q}$

for $\alpha \in B_{l,m-l}$ is in the thick subcategory $\mathcal{C}$ generated by $\mathcal{T}$. Inductively, we may assume that α is such that $L^\beta \mathcal{Q}$ is in $\mathcal{C}$ for all β less than α in the lexicographic ordering on partitions.

Consider $\mathcal{U} = \bigwedge^{\alpha'_1} \mathcal{Q} \otimes_{\mathbb{G}} \cdots \otimes_{\mathbb{G}} \bigwedge^{\alpha'_l} \mathcal{Q}$. Then Pieri's formula, which is a special case of the Littlewood–Richardson rule, yields a filtration of $\mathcal{U}$ with successive quotients $L^\beta \mathcal{Q}$ such that $\beta \leqslant \alpha$ and such that $L^\alpha \mathcal{Q}$ appears with multiplicity one. Furthermore, $\mathcal{U}$ has a *good filtration* [Jan03, § II.4.16], one in which the $L^\beta \mathcal{Q}$ appearing as quotients are in decreasing order for the lexicographic ordering on partitions, i.e. the largest β appear on top [Jan03, II.4.16, Remark (4)]. Hence $\mathcal{U}$ maps surjectively to $L^\alpha \mathcal{Q}$, and the kernel is an extension of various $L^\beta \mathcal{Q}$ with β strictly smaller than α in the lexicographic ordering. By the hypothesis all such $L^\beta \mathcal{Q}$ are in $\mathcal{C}$. Since $\mathcal{U}$ is in $\mathcal{C}$ as well we obtain that $L^\alpha \mathcal{Q}$ is in $\mathcal{C}$. □

Remark 3.1. By [Don93, Lemma (3.4)] the indecomposable summands of $\bigwedge^{\alpha'} V$ are precisely the *tilting representations* for $\mathrm{GL}(V)$, so we can obtain the following more economical tilting bundle for $\mathbb{G}$:

$$\mathcal{T}^\circ = \bigoplus_{\alpha \in B_{l,m-l}} \mathcal{L}_{\mathbb{G}}(M^\alpha),$$

where M^α is the tilting $\mathrm{GL}(l)$-representation with highest weight α [Jan03, E.3]. Note, however, that the character of M^α strongly depends on the characteristic. Hence so does the nature of $\mathcal{T}^\circ$.

For use below we need the following complement to Proposition 1.4.

Proposition 3.2. *For every partition $\alpha \in B_{l,m-l}$ and every polynomial G_1-representation U of degree less than $|\alpha|$, we have*

$$\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(\textstyle\bigwedge^{\alpha'} \mathcal{Q}, \mathcal{L}_{\mathbb{G}}(U)) = 0. \tag{3.7}$$

Proof. It suffices to prove the claimed vanishing for U simple of degree less than $|\alpha|$, so for $U = \Sigma^\beta$ with β a partition such that $|\beta| < |\alpha|$. We do this by induction on β. Since Σ^β is the socle of $L^\beta \mathcal{Q}$ we have, by [Jan03, Proposition 6.15], a short exact sequence

$$0 \longrightarrow \Sigma^\beta \longrightarrow L^\beta V \longrightarrow S \longrightarrow 0$$

where S is obtained through extensions involving only Σ^γ with $\gamma < \beta$. By induction we may assume $\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'} \mathcal{Q}, \mathcal{L}_{\mathbb{G}}(S)) = 0$. Then (3.7) for $U = \Sigma^\beta$ follows from Proposition 1.4. □

4. Reminder on semi-orthogonal decompositions

We recapitulate some facts concerning semi-orthogonal decompositions that we need later. No originality is intended.

If $\mathcal{S}$ is a triangulated category and S is a set of objects then we denote by $\langle S \rangle$ the smallest triangulated subcategory of $\mathcal{S}$ that contains S and is closed under isomorphisms. If $\mathcal{S} = \langle S \rangle$ then we say that S generates $\mathcal{S}$ as a triangulated category. (This is stronger than 'classically' generating $\mathcal{S}$ as in the Introduction.)

Definition 4.1. A *semi-orthogonal decomposition* of a triangulated category $\mathcal{S}$ is a sequence of full subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_n \subset \mathcal{S}$ generating $\mathcal{S}$ as a triangulated category and such that $\mathrm{Hom}_{\mathcal{S}}(\mathcal{A}_j, \mathcal{A}_i) = 0$ for $j > i$. We denote such a semi-orthogonal decomposition by $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$. Sometimes we write $\mathcal{S} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$.

If X is an object in a triangulated category then a filtration F of length n on X is a sequence of maps

$$0 = F_n X \longrightarrow F_{n-1} X \longrightarrow \cdots \longrightarrow F_0 X = X.$$

We write $(\operatorname{gr}_F X)_i = \mathsf{cone}(F_{i+1}X \longrightarrow F_i X)$. The following well-known lemma shows that Definition 4.1 is equivalent to the seemingly stronger one in [Kuz09, Definition 2.3].

Lemma 4.2. *Let $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ be a semi-orthogonal decomposition of $\mathcal{S}$. Then every object X in $\mathcal{S}$ has a filtration F of length n such that $(\operatorname{gr}_F X)_i \in \mathcal{A}_{i+1}$.*

Proof. By induction it is sufficient to prove this for $n = 2$. In that case the result is [Bon89, Lemma 3.1]. □

In order to work conveniently with semi-orthogonal decompositions one needs a property called *admissibility* [BK89]. If $\mathcal{A}$ is a full triangulated subcategory of a triangulated category $\mathcal{S}$ then $\mathcal{A}$ is (left, right) admissible if the inclusion functor $\mathcal{A} \longrightarrow \mathcal{S}$ has a (left, right) adjoint, or equivalently if there exist semi-orthogonal decompositions $\langle \mathcal{A}, \mathcal{A}' \rangle$ (respectively, $\langle \mathcal{A}'', \mathcal{A} \rangle$). If $\mathcal{A}$ is both left and right admissible then it is said to be admissible.

A *saturated* triangulated category is a K-linear triangulated category $\mathcal{A}$ such that, for all A, $B \in \mathcal{A}$, we have $\sum_i \dim \operatorname{Hom}^i_{\mathcal{A}}(A, B) < \infty$ and such that every co- or contravariant cohomological functor $H^i : \mathcal{A} \longrightarrow \operatorname{mod}(K)$ satisfying $\sum_i \dim H^i(A) < \infty$ is representable. The derived category of coherent sheaves on a smooth proper algebraic variety is a particular example of a saturated triangulated category [BVdB03, BK89].

If $\mathcal{A}$ is a saturated full triangulated subcategory of a K-linear triangulated category $\mathcal{S}$ then $\mathcal{A}$ is admissible [BK89, Proposition 2.6]. Furthermore, if $\mathcal{S}$ is a saturated triangulated category then every left/right admissible subcategory in $\mathcal{S}$ is automatically admissible (and hence saturated). This follows by combining [BK89, Propositions 2.6 and 2.8]. From this we deduce that if we have a semi-orthogonal decomposition $\mathcal{S} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ of a saturated $\mathcal{S}$ then all the 'slices' $\langle \mathcal{A}_i, \ldots, \mathcal{A}_j \rangle$ are admissible and saturated.

In particular, if we put $\mathcal{S}_{\leqslant i} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_i \rangle$ then this yields a filtration $\mathcal{S}_{\leqslant 1} \subset \cdots \subset \mathcal{S}_{\leqslant n} = \mathcal{S}$ by admissible subcategories. Let $\mathcal{B}_i$ be the right orthogonal of $\mathcal{S}_{\leqslant i-1}$ in $\mathcal{S}_{\leqslant i}$, i.e. the full subcategory of objects B in $\mathcal{S}_{\leqslant i}$ such that $\operatorname{Hom}_{\mathcal{S}_{\leqslant i}}(A, B) = 0$ for every object A in $\mathcal{S}_{\leqslant i-1}$. Then we have semi-orthogonal decompositions $\mathcal{S}_{\leqslant i} = \langle \mathcal{B}_i, \mathcal{S}_{\leqslant i-1} \rangle$. Iterating, we obtain a semi-orthogonal decomposition

$$\mathcal{S} = \langle \mathcal{B}_n, \mathcal{B}_{n-1}, \ldots, \mathcal{B}_1 \rangle$$

such that

$$\langle \mathcal{A}_1, \ldots, \mathcal{A}_i \rangle = \langle \mathcal{B}_i, \ldots, \mathcal{B}_1 \rangle.$$

This is called the semi-orthogonal decomposition (right) dual to $\mathcal{S} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$. Note that [Kuz09, (4)]

$$\begin{aligned}
\mathcal{B}_i &= \mathcal{S}_{\leqslant i} \cap \mathcal{S}_{\leqslant i-1}^{\perp} \\
&= \langle \mathcal{A}_{i+1}, \ldots, \mathcal{A}_n \rangle^{\perp} \cap \langle \mathcal{A}_1, \ldots, \mathcal{A}_{i-1} \rangle^{\perp} \\
&= \langle \mathcal{A}_1, \ldots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1}, \ldots, \mathcal{A}_n \rangle^{\perp},
\end{aligned}$$

where $-^{\perp}$ indicates the right orthogonal in $\mathcal{S}$. In particular, for $i \neq j$,

$$\operatorname{Hom}(\mathcal{A}_i, \mathcal{B}_j) = 0. \tag{4.1}$$

The following is also well known [Kuz09].

Lemma 4.3. *Assume $\mathcal{S}$ is saturated. Let γ_i be the composition of the canonical functors*

$$\gamma_i : \mathcal{A}_i \longrightarrow \mathcal{S}_{\leqslant i} \longrightarrow \mathcal{S}_{\leqslant i}/\mathcal{S}_{\leqslant i-1} = \mathcal{B}_i.$$

Then γ_i is an equivalence of categories. Furthermore, we have for $A \in \mathcal{A}_i$, $B \in \mathcal{B}_i$,

$$\operatorname{Hom}_{\mathcal{S}}(A, B) = \operatorname{Hom}_{\mathcal{B}_i}(\gamma_i(A), B) = \operatorname{Hom}_{\mathcal{S}}(\gamma_i(A), B).$$

Proof. We have semi-orthogonal decompositions

$$\mathcal{S}_{\leqslant i} = \langle \mathcal{S}_{\leqslant i-1}, \mathcal{A}_i \rangle = \langle \mathcal{B}_i, \mathcal{S}_{\leqslant i-1} \rangle.$$

The fact that γ_i is an equivalence follows from [BK89, Lemma 1.9].

Let $\jmath : \mathcal{A}_i \longrightarrow \mathcal{S}_{\leqslant i}$ and $\imath : \mathcal{B}_i \longrightarrow \mathcal{S}_{\leqslant i}$ be the inclusion functors and let $\imath^*$ be the left adjoint to $\imath$. Then $\gamma_i = \imath^* \circ \jmath$. We have

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{S}}(A, B) &= \operatorname{Hom}_{\mathcal{S}_{\leqslant i}}(\jmath A, \imath B) \\
&= \operatorname{Hom}_{\mathcal{B}_i}(\imath^* \jmath A, B) \\
&= \operatorname{Hom}_{\mathcal{B}_i}(\gamma_i(A), B).
\end{aligned}$$

The equality $\operatorname{Hom}_{\mathcal{B}_i}(\gamma_i(A), B) = \operatorname{Hom}_{\mathcal{S}}(\gamma_i(A), B)$ is just that $\mathcal{B}_i$ is a full subcategory of $\mathcal{S}$. □

5. Semi-orthogonal decompositions for Grassmannians

In this section we write $\mathcal{D}$ for the bounded derived category of coherent sheaves on $\mathbb{G}$. This is, in particular, a saturated category (see § 4). We will construct a semi-orthogonal decomposition of $\mathcal{D}$.

We start by observing that the proof of Theorem 1.3 actually shows the following result.

Lemma 5.1. *$\mathcal{D}$ is generated as a triangulated category by $(\bigwedge^{\alpha'} \mathcal{Q})_{\alpha \in B_{l,m-l}}$ (instead of just classically generated; see § 4).*

A set S of dominant weights is *saturated* if, whenever $\alpha \in S$ and $\beta < \alpha$ is dominant, we have $\beta \in S$. (Here and below '$<$' is the standard partial ordering on weights; see § 2. In particular, note that $\beta < \alpha$ implies $|\beta| = |\alpha|$.) The set $B_{l,m-l}$ is an example of a saturated set for $\mathrm{GL}(l)$. For $d \geqslant 0$, let $\mathcal{C}_d$ be the category of finite-dimensional $G_1 = \mathrm{GL}(l)$-representations whose composition factors have highest weights α satisfying $|\alpha| = d$ and $\alpha \in B_{l,m-l}$. Thus $\mathcal{C}_d$ is a *truncated category* in the sense of [Jan03, ch. A] associated to a saturated set of dominant weights. In particular, $\mathcal{C}_d$ is the category of finite modules over a certain finite-dimensional algebra, called a *generalized Schur algebra* [Jan03, § A.16].

We collect some elementary facts about the derived category of $\mathcal{C}_d$.

Lemma 5.2. *Let $\operatorname{Rep}(G_1)$ be the category of rational G_1-representations, and for each d let $D^b_{\mathcal{C}_d}(\operatorname{Rep}(G_1))$ be the bounded derived category of complexes of representations having cohomology in $\mathcal{C}_d$. The canonical functor*

$$D^b(\mathcal{C}_d) \longrightarrow D^b_{\mathcal{C}_d}(\operatorname{Rep}(G_1))$$

is an equivalence of categories.

Proof. That the functor is fully faithful follows from the fact that the Yoneda Exts in $\mathcal{C}_d$ are the same as those in the ambient category $\mathrm{Rep}(G_1)$ (see [Jan03, Proposition A.10]). Essential surjectivity follows from full faithfulness and the fact that the essential image contains the generating subcategory $\mathcal{C}_d$. □

In the sequel we will simply confuse $D^b(\mathcal{C}_d)$ and $D^b_{\mathcal{C}_d}(\mathrm{Rep}(G_1))$.

Lemma 5.3. *The triangulated category $D^b(\mathcal{C}_d)$ is generated by the representations $\bigwedge^{\alpha'} V$ for $\alpha \in B_{l,m-l}$, $|\alpha| = d$, where, as usual, V is the standard representation of G_1.*

Proof. This is of course well known, but for the convenience of the reader we give the proof. Let $\mathcal{A}$ be the full subcategory of $D^b(\mathcal{C}_d)$ generated by $(\bigwedge^{\alpha'} V)_{\alpha \in B_{l,m-l}, |\alpha|=d}$. It is sufficient to prove that $\mathcal{A}$ contains the simple modules Σ^α for $\alpha \in B_{l,m-l}$, $|\alpha| = d$.

We reason as in the proof of Theorem 1.3: for $\alpha \in B_{l,m-l}$ with $|\alpha| = d$, the representation $\mathcal{U} = \bigwedge^{\alpha'_1} V \otimes \cdots \otimes \bigwedge^{\alpha'_l} V$ has a *Weyl filtration* [Jan03, II.4.19], i.e. one in which the successive quotients are of the form $K^\beta V$ for $\beta \leqslant \alpha$, with $K^\alpha V$ appearing (with multiplicity one) as the first term. Then the cokernel of the inclusion $K^\alpha V \hookrightarrow \mathcal{U}$ is in $\mathcal{A}$ by induction, whence $K^\alpha V$ is in $\mathcal{A}$ as well. (We point out that it is essential to work with Weyl filtrations rather than good filtrations here, since $B_{l,m-l}$ is only saturated, not cosaturated.)

By [Jan03, II.2.13], $K^\alpha V$ has simple top Σ^α, and by the dual version of [Jan03, II.6.13] the other Jordan–Hölder quotients of $K^\alpha V$ are of the form Σ^γ with $|\gamma| = |\alpha| = d$ and $\gamma < \alpha$. Thus $\Sigma^\gamma \in \mathcal{C}_d$. By induction we may assume that such $\Sigma^\gamma \in \mathcal{A}$. Hence it follows that $\Sigma^\alpha \in \mathcal{A}$. □

We define a functor
$$\Phi_d : D^b(\mathcal{C}_d) \longrightarrow \mathcal{D}$$
by $\Phi_d(U) = \mathcal{L}_{\mathbb{G}}(U)$ for $U \in D^b(\mathcal{C}_d)$, where we view U as a complex of P-representations in the obvious way. Observe that Φ_d is a monoidal functor and takes V to $\mathcal{Q}$, hence, in particular, $\Phi_d(\bigwedge^{\alpha'} V) = \bigwedge^{\alpha'} \mathcal{Q}$.

Theorem 5.4. *The functor Φ_d is fully faithful.*

Proof. By Lemma 5.3 it is sufficient to prove that, for $\alpha, \beta \in B_{l,m-l}$ with $|\alpha| = |\beta| = d$, the canonical map
$$\mathrm{RHom}_{G_1}(\textstyle\bigwedge^{\alpha'} V, \bigwedge^{\beta'} V) \longrightarrow \mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'} \mathcal{Q}, \bigwedge^{\beta'} \mathcal{Q}) \tag{5.1}$$
is an isomorphism (where we have used that $\mathcal{L}_{\mathbb{G}}(V) = \mathcal{Q}$). Now $\bigwedge^{\alpha'} V$ and $\bigwedge^{\beta'} V$ are tilting representations [Don93, Lemma (3.4)], and so on the left-hand side of (5.1) there are no higher Exts. Likewise on the right-hand side there are no higher Exts because of Proposition 1.4.

So we only have to show that the map
$$\mathrm{Hom}_{G_1}(\textstyle\bigwedge^{\alpha'} V, \bigwedge^{\beta'} V) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'} \mathcal{Q}, \bigwedge^{\beta'} \mathcal{Q})$$
is an isomorphism. It is certainly injective: take the fiber of the right-hand side in $[P] \in G/P$ to get $\mathrm{Hom}(\bigwedge^{\alpha'} V, \bigwedge^{\beta'} V)$, and the composition is injective. Thus we need only compute the K-dimensions of each side.

The filtrations of $\bigwedge^{\beta'} V$ and $\bigwedge^{\beta'} \mathcal{Q}$ by $L^\gamma V$ and $L^\gamma \mathcal{Q}$, respectively, and the vanishing of the higher Exts from Proposition 1.4, reduce the problem to that of computing K-dimensions of $\mathrm{Hom}_{\mathrm{GL}(l)}(\bigwedge^{\alpha'} V, L^\beta V)$ and $\mathrm{Hom}_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'} \mathcal{Q}, L^\beta \mathcal{Q})$. Both dimensions are, by the comment following Theorem 2.3 and Proposition 1.4, equal to the number of semi-standard tableaux of shape β' and content α'. □

Now let $\mathcal{D}_d$ be the essential image of $D^b(\mathcal{C}_d)$ under Φ_d, i.e. the closure of that image under isomorphisms. From Lemma 5.3 we obtain the following corollary.

Corollary 5.5. *$\mathcal{D}_d$ is generated by $\bigwedge^{\alpha'}\mathcal{Q}$ for $\alpha \in B_{l,m-l}$, $|\alpha| = d$.*

We have the following theorem.

Theorem 5.6. *The triangulated category $\mathcal{D}$ has a semi-orthogonal decomposition*

$$\mathcal{D} = \langle \mathcal{D}_0, \dots, \mathcal{D}_{l(m-l)} \rangle. \tag{5.2}$$

Furthermore, $\mathcal{D}_d$ is generated by $L^\alpha \mathcal{Q}$ for $\alpha \in B_{l,m-l}$ with $|\alpha| = d$.

Proof. By Corollary 5.5, $\mathcal{D}_d$ is generated by those $\bigwedge^{\alpha'}\mathcal{Q}$ with $|\alpha| = d$, so it follows from Lemma 5.1 that $\mathcal{D}$ is generated by $(\mathcal{D}_d)_d$.

To complete the proof that (5.2) is a semi-orthogonal decomposition we need that $\operatorname{Hom}(\mathcal{D}_d, \mathcal{D}_e) = 0$ for $d > e$, or equivalently that $\operatorname{RHom}_{\mathcal{D}}(\bigwedge^{\alpha'}\mathcal{Q}, \bigwedge^{\beta'}\mathcal{Q}) = 0$ for $|\alpha| = d$, $|\beta| = e$. This follows from Proposition 3.2. $\square$

The theorems we have stated have dual versions where $\mathcal{Q}$ is replaced by $\mathcal{R}$ and $B_{l,m-l}$ by $B_{m-l,l}$. We prove these by passing to the dual Grassmannian $\operatorname{Grass}(m-l, F^\vee)$.

Lemma 5.7. *The vector bundle*

$$\mathcal{T}' = \bigoplus_{\alpha \in B_{m-l,l}} \bigwedge\nolimits^{\alpha'}\mathcal{R} \tag{5.3}$$

is a tilting bundle on $\mathbb{G}$.

Proof. Using the duality $\mathrm{R}\mathcal{H}om_{\mathcal{O}_{\mathbb{G}}}(-, \mathcal{O}_{\mathbb{G}})$ on $\mathcal{D}$, it is sufficient to show that $\mathcal{T}'^\vee$ is a tilting bundle. Now $\mathcal{T}'^\vee$ is equal to $\bigoplus_{\alpha \in B_{m-l,l}} \bigwedge^{\alpha'}(\mathcal{R}^\vee)$ and we see that the latter is a tilting bundle by passing to the dual Grassmannian (which replaces $\mathcal{R}^\vee$ by $\mathcal{Q}$) and invoking Theorem 1.3. $\square$

For $d \geqslant 0$, let $\mathcal{C}'_d$ be the category of finite-dimensional $G_2 = \operatorname{GL}(m-l)$-representations whose composition factors have highest weights α satisfying $|\alpha| = d$ and $\alpha \in B_{m-l,l}$. We have the following analogue of Theorem 5.4, where

$$\Phi'_d : D^b(\mathcal{C}'_d) \longrightarrow \mathcal{D}$$

is defined again by $U \mapsto \mathcal{L}_{\mathbb{G}}(U)$.

Theorem 5.8. *The functor Φ'_d is fully faithful.*

Proof. This follows by dualizing the proof of Theorem 5.4. $\square$

Below we let $\mathcal{D}'_d$ be the essential image of $D^b(\mathcal{C}'_d)$ under Φ'_d. We obtain the following analogue of Corollary 5.5.

Lemma 5.9. *$\mathcal{D}'_d$ is generated by $\bigwedge^{\alpha'}\mathcal{R}$ for $\alpha \in B_{m-l,l}$, $|\alpha| = d$.*

Theorem 5.10. *The triangulated category $\mathcal{D}$ has a semi-orthogonal decomposition*

$$\langle \mathcal{D}'_{l(m-l)}, \dots, \mathcal{D}'_0 \rangle.$$

Furthermore, $\mathcal{D}'_d$ is generated by $K^\alpha \mathcal{R}$ for $\alpha \in B_{m-l,l}$ with $|\alpha| = d$.

Proof. This follows by dualizing the proof of Theorem 5.6. ☐

Proposition 5.11. *Let $\operatorname{rep}_e(G_i)$ be the category of finite-dimensional G_i-representations of degree $e \geqslant 0$. Then $\mathcal{L}_{\mathbb{G}}(U)$, for $U \in \operatorname{rep}_e(G_i)$ with $e \leqslant d$, is contained in $\langle \mathcal{D}_e \rangle_{e \leqslant d}$ if $i = 1$ and $\langle \mathcal{D}'_e \rangle_{e \leqslant d}$ if $i = 2$.*

Note that we do not assume that the dominant weights of U are in $B_{l,m-l}$ or $B_{m-l,l}$. For example the proposition applies to $\alpha = [e, 0, \ldots, 0]$ with $e > m - l$.

Proof. To prove this for $i = 1$ we have to show that $\operatorname{Hom}(\mathcal{D}_f, \mathcal{L}_{\mathbb{G}}(U)) = 0$ for $f > d$. Given that $\mathcal{D}_f$ is generated by $\bigwedge^{\alpha'} \mathcal{Q}$ for $\alpha \in B_{l,m-l}$ and $|\alpha| = f$, this follows from Proposition 3.2. The argument for $i = 2$ is dual. ☐

The following result finishes this section.

Theorem 5.12. *The semi-orthogonal decompositions*

$$\mathcal{D} = \langle \mathcal{D}_0, \ldots, \mathcal{D}_{l(m-l)} \rangle \quad \text{and} \quad \mathcal{D} = \langle \mathcal{D}'_{l(m-l)}, \ldots, \mathcal{D}'_0 \rangle$$

are dual to each other. Furthermore, the induced equivalence $\gamma_d : \mathcal{D}_d \longrightarrow \mathcal{D}'_d$ defined in Lemma 4.3 sends $L^\alpha \mathcal{Q}$ to $K^{\alpha'} \mathcal{R}[d]$ for $\alpha \in B_{l,m-l}$ with $|\alpha| = d$.

Proof. To prove that the semi-orthogonal decompositions are dual, according to § 4 we have to show that

$$\mathcal{D}_{\leqslant d} = \mathcal{D}'_{\leqslant d},$$

where we set $\mathcal{D}_{\leqslant d} = \langle \mathcal{D}_0, \ldots, \mathcal{D}_d \rangle$ and $\mathcal{D}'_{\leqslant d} = \langle \mathcal{D}'_d, \ldots, \mathcal{D}'_0 \rangle$. We prove the inclusion $\mathcal{D}_{\leqslant d} \subset \mathcal{D}'_{\leqslant d}$. The opposite inclusion is similar.

From Theorem 5.6 we obtain that $\mathcal{D}_{\leqslant d}$ is generated by $L^\alpha \mathcal{Q}$ for $|\alpha| \leqslant d$. Thus we have to show that for such α we have $L^\alpha \mathcal{Q} \in \mathcal{D}'_{\leqslant d}$.

According to [Wey03, ch. 2, Example 21], we have a resolution for $L^\alpha \mathcal{Q}$ given by the Schur complex

$$L^\alpha(\mathcal{R} \longrightarrow F^\vee \otimes \mathcal{O}_{\mathbb{G}})$$

and, furthermore, by [Wey03, Theorem (2.4.10)(b)], $L^\alpha(\mathcal{R} \longrightarrow F^\vee \otimes \mathcal{O}_{\mathbb{G}})$ has a filtration with associated graded object

$$\operatorname{gr} L^\alpha(\mathcal{R} \longrightarrow F^\vee \otimes \mathcal{O}_{\mathbb{G}})_t = \bigoplus_{\substack{|\nu| = t \\ \nu \subset \alpha}} K^{\nu'} \mathcal{R} \otimes L^{\alpha/\nu}(F^\vee). \tag{5.4}$$

By Proposition 5.11, all $K^{\nu'} \mathcal{R}$ are in $\mathcal{D}'_{\leqslant d}$. Hence so is $L^\alpha Q$.

Assume now $|\alpha| = d$. In that case (5.4) shows that

$$L^\alpha \mathcal{Q} = K^{\alpha'} \mathcal{R}[|\alpha|] \mod \mathcal{D}'_{\leqslant d-1}.$$

If, in addition, $\alpha \in B_{l,m-l}$ then Lemma 5.9 implies $K^{\alpha'} \mathcal{R}[|\alpha|] \in \mathcal{D}'_d$, from which we conclude that $\gamma_d(L^\alpha \mathcal{Q}) = K^{\alpha'} \mathcal{R}[|\alpha|]$. ☐

6. Some more comments on representation theory

If we combine Theorems 5.4, 5.8, and 5.12 we obtain an equivalence of categories $\delta_d \overset{\text{def}}{=} (\Phi_d'^{-1} \circ \gamma_d \circ \Phi_d)[-d]$ between $D^b(\mathcal{C}_d)$ and $D^b(\mathcal{C}'_d)$. The existence of such an equivalence is well known (see, for example, [Don93, Corollary (3.9)] for a similar result) but the standard construction uses the representation theory of the symmetric group.

Below we list some properties of the equivalence, which we will use in § 8. For α a partition, let M^α be the indecomposable tilting G_1-representation with highest weight α. Similarly, for $\beta \in B_{m-l,l}$, let Σ'^β be the simple G_2-representation with highest weight β and let P'^β be the projective cover of Σ'^β in $\mathcal{C}'_d$.

Since M^α has highest weight α and since $B_{l,m-l}$ is a saturated set of partitions, all the dominant weights of M^α are in $B_{l,m-l}$, whence $M^\alpha \in \mathcal{C}_d$ by [Jan03, Lemma E.3].

Proposition 6.1. *We have, for $\alpha \in B_{l,m-l}$ with $|\alpha| = d$,*

$$\delta_d(L^\alpha V) = K^{\alpha'} W, \tag{6.1}$$

$$\delta_d(M^\alpha) = P'^{\alpha'}. \tag{6.2}$$

Proof. Statement (6.1) follows from Theorem 5.12. To prove (6.2) we first note that since M_α is a tilting representation, it has a good filtration, i.e. one with successive quotients $L^\beta V$, and hence (6.1) implies $\delta_d(M^\alpha) \in \mathcal{C}'_d$. Furthermore, since δ_d is an equivalence, for $i > 0$ and $\beta \in B_{l,m-l}$ with $|\beta| = d$ we have

$$\mathrm{Ext}^i_{G_2}(\delta_d(M^\alpha), K^{\beta'} W) = \mathrm{Ext}^i_{G_2}(\delta_d(M^\alpha), \delta_d(L^\beta V)) = \mathrm{Ext}^i_{G_1}(M^\alpha, L^\beta V) = 0. \tag{6.3}$$

We now claim that, for $i > 0$,

$$\mathrm{Ext}^i_{G_2}(\delta_d(M^\alpha), \Sigma^{\beta'}) = 0. \tag{6.4}$$

We prove this by induction. Dually to Proposition 3.2, we have a short exact sequence

$$0 \longrightarrow U \longrightarrow K^{\beta'} W \longrightarrow \Sigma^{\beta'} \longrightarrow 0$$

where U is obtained through extensions involving only Σ^γ with $\gamma < \beta'$. By induction we may assume $\mathrm{Ext}^i_{G_2}(\delta_d(M^\alpha), U) = 0$. Then (6.4) follows from (6.3). We conclude that $\delta_d(M^\alpha)$ is projective. Since M^α is indecomposable, the same is true for $\delta_d(M^\alpha)$. Hence $\delta_d(M^\alpha)$ is equal to some P'^γ. To prove that $\delta_d(M^\alpha) = P'^{\alpha'}$ it is sufficient to construct a surjective map

$$\delta_d(M^\alpha) \longrightarrow K^{\alpha'} W$$

since $K^{\alpha'} W$ has simple top $\Sigma'^{\alpha'}$.

By [Jan03, § E.4], we have a short exact sequence

$$0 \longrightarrow H \longrightarrow M^\alpha \longrightarrow L^\alpha V \longrightarrow 0$$

where H is an extension of $L^\gamma W$ with $\gamma < \alpha$. After applying δ_d, this becomes a distinguished triangle

$$\delta_d(H) \longrightarrow \delta_d(M^\alpha) \longrightarrow K^{\alpha'} W \longrightarrow$$

with $\delta_d(H), \delta_d(M^\alpha) \in \mathcal{C}'_d$. The long exact sequence for cohomology shows that $\delta_d(M^\alpha) \longrightarrow K^{\alpha'} W$ is indeed surjective. □

7. Exceptional sequences on Grassmannians

PROPOSITION 7.1. *Assume $\alpha, \beta \in B_{l,m-l}$ with $|\alpha| = |\beta|$. If*

$$\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha \mathcal{Q}, L^\beta \mathcal{Q}) \neq 0 \tag{7.1}$$

then $\alpha \geqslant \beta$. Furthermore,

$$\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha \mathcal{Q}, L^\alpha \mathcal{Q}) = K.$$

Proof. If $\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha \mathcal{Q}, L^\beta \mathcal{Q}) \neq 0$ then it follows that $\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'} \mathcal{Q}, L^\beta \mathcal{Q}) \neq 0$ since, as in the proof of Theorem 1.3, $\bigwedge^{\alpha'} \mathcal{Q}$ has a filtration with successive quotients $L^\gamma \mathcal{Q}$ such that $\gamma \leqslant \alpha$ and $L^\alpha \mathcal{Q}$ appears with multiplicity one. By Proposition 1.4, $\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(\bigwedge^{\alpha'} \mathcal{Q}, L^\beta \mathcal{Q})$ is concentrated in (at most) a single degree, and has K-dimension equal to the number of semi-standard tableaux of shape l^{m-l}/β' and content $l^{m-l} - \alpha'$. In particular, this number is zero if $\alpha < \beta$ in the natural ordering on partitions, so we must have $\alpha \geqslant \beta$.

The additional statement follows from the first and Proposition 1.4, using once more the good filtration of $\bigwedge^{\alpha'} \mathcal{Q}$ and the fact that there is exactly one semi-standard tableau of the type required. □

PROPOSITION 7.2. *Assume $\alpha, \beta \in B_{m-l,l}$, $|\alpha| = |\beta|$. Then*

$$\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha \mathcal{R}, L^\beta \mathcal{R}) \neq 0 \tag{7.2}$$

implies $\alpha \geqslant \beta$. Furthermore,

$$\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha \mathcal{R}, L^\alpha \mathcal{R}) = K.$$

Proof. This is proved by passing to the dual Grassmannian and applying Proposition 7.1. □

COROLLARY 7.3. *The subcollections*

$$(L^\alpha \mathcal{Q})_{\alpha \in B_{l,m-l},\ |\alpha| = d} \quad \text{and} \quad (L^\alpha \mathcal{R})_{\alpha \in B_{m-l,l},\ |\alpha| = d}$$

form exceptional collections in $\mathcal{D}_d$.

PROPOSITION 7.4. *For $\alpha, \beta \in B_{m-l,l}$, $|\alpha| = |\beta|$, we have*

$$\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(K^\alpha \mathcal{R}, L^\beta \mathcal{R}) = \delta_{\alpha,\beta} \cdot K.$$

Proof. Put $d = |\alpha| = |\beta|$. We have $K^\alpha \mathcal{R} = \Phi'_d(K^\alpha W)$, $L^\beta \mathcal{R} = \Phi'_d(L^\beta W)$. So by Theorem 5.8, we must show

$$\mathrm{RHom}_{G_2}(K^\alpha W, L^\beta W) = \delta_{\alpha,\beta} \cdot K.$$

As $K^\alpha W$ is a Weyl representation and $L^\beta W$ is an induced representation, it suffices to invoke [Jan03, II.4.13]. □

Now we make $B_{l,m-l}$ into a totally ordered set by equipping it with an arbitrary total ordering $\prec$ such that if $|\alpha| < |\beta|$ then $\alpha \prec \beta$, and if $|\alpha| = |\beta|$ and $\alpha > \beta$ in the standard partial order on partitions then $\alpha \prec \beta$. We write $\overline{B}_{l,m-l}$ for $B_{l,m-l}$, equipped with the opposite ordering.

The following is the main result of this section.

THEOREM 7.5. *The collections $(L^\alpha \mathcal{Q})_{\alpha \in B_{l,m-l}}$ and $(L^{\beta'} \mathcal{R}[|\beta|])_{\beta \in \overline{B}_{l,m-l}}$ form dual exceptional collections in $\mathcal{D}$. In other words, for $\alpha, \beta \in B_{l,m-l}$, we have*

$$\mathrm{RHom}_{\mathcal{O}_{\mathbb{G}}}(L^\alpha \mathcal{Q}, L^{\beta'} \mathcal{R}[|\beta|]) = \delta_{\alpha,\beta} \cdot K. \tag{7.3}$$

Proof. The fact that $(L^\alpha\mathcal{Q})_{\alpha\in B_{l,m-l}}$ is an exceptional sequence follows from Theorem 5.6 and Proposition 7.1. Similarly, the fact that $(L^{\beta'}\mathcal{R}[|\beta|])_{\beta\in\overline{B}_{l,m-l}}$ is an exceptional collection follows from Theorem 5.10 and Proposition 7.2. So it remains to prove the duality property (7.3). By Theorem 5.12 combined with (4.1), we may assume that $|\alpha|=|\beta|$. We compute, for all $i\in\mathbb{Z}$,

$$\begin{aligned}\operatorname{Ext}^i_{\mathcal{O}_\mathbb{G}}(L^\alpha\mathcal{Q},L^{\beta'}\mathcal{R}[|\beta|]) &= \operatorname{Ext}^i_{\mathcal{O}_\mathbb{G}}(\gamma_{|\alpha|}(L^\alpha\mathcal{Q}),L^{\beta'}\mathcal{R}[|\beta|])\\ &= \operatorname{Ext}^i_{\mathcal{O}_\mathbb{G}}(K^{\alpha'}\mathcal{R}[|\alpha|],L^{\beta'}\mathcal{R}[|\beta|])\\ &= \delta_{\alpha,\beta}\cdot\delta_{i0}\cdot K,\end{aligned}$$

using Lemma 4.3, Theorem 5.12, and Proposition 7.4, respectively. □

To conclude this section we use the 'linkage principle' to recover the result of Kaneda, mentioned in the Introduction, that Kapranov's tilting bundle $\mathcal{K}$ remains tilting in large characteristic. For the convenience of the reader we state the linkage principle in the case of interest to us. Recall that Σ^γ denotes the simple socle of $L^\gamma V$.

Theorem 7.6 ([Jan03, Corollary II.6.17]). *If γ, δ are dominant weights for G_1 and $\operatorname{Ext}^1_{G_1}(\Sigma^\gamma,\Sigma^\delta)\neq 0$ then γ, δ are in the same orbit for the* affine Weyl group.

Lemma 7.7. *Assume that K has characteristic p with $p\geqslant m-1$. Let $\alpha\in B_{l,m-l}$. Then $\bigwedge^{\alpha'}\mathcal{Q}$ is a direct sum of $L^\beta\mathcal{Q}$ with $|\beta|=|\alpha|$ and, furthermore, there are no homomorphisms between the summands of $\bigwedge^{\alpha'}\mathcal{Q}$.*

Proof. Set $d=|\alpha|$. Using Theorem 5.4, it is enough to prove the following claim: $\mathcal{C}_d$ is a semi-simple category with simple objects given by $L^\beta V$ for $\beta\in B_{l,m-l}$ with $|\beta|=d$. Indeed, if this claim holds then $\bigwedge^{\alpha'}V$ is a direct sum of the simple objects $L^\beta V$ (which thus have no Homs among them) and it suffices to apply the fully faithful functor $\Phi(-)_d$ to obtain the corresponding result for $\bigwedge^{\alpha'}\mathcal{Q}$.

To derive the claim from the linkage principle, recall that a fundamental domain[2] $\overline{C}$ for the affine Weyl group [Jan03, II.6.1(6)] is given by the set of weights $x=[x_1,\dots,x_l]$ satisfying $0\leqslant\langle x+\rho,\alpha\rangle\leqslant p$ for all positive roots α, where $\rho=[l,l-1,\dots,1]$; equivalently,

$$0\leqslant x_i-i-x_j+j\leqslant p$$

for $j>i$. The first inequality is automatically satisfied for a dominant weight. For the second inequality we note that if $\gamma=[\gamma_1,\dots,\gamma_l]\in B_{l,m-l}$ then

$$\gamma_i-\gamma_j\leqslant m-l$$

and

$$-i+j\leqslant l-1.$$

Thus

$$\gamma_i-i-\gamma_j+j\leqslant m-l+l-1=m-1\leqslant p.$$

In other words $B_{l,m-l}\subseteq\overline{C}$ and thus no two elements of $B_{l,m-l}$ are in the same orbit for the affine Weyl group.

Since $L^\alpha V$ has a filtration with factors being Σ^β with $\beta\in B_{l,m-l}$ and since there are no non-trivial extensions between them, it follows that $L^\alpha V$ is a direct sum of Σ^β. On the other hand, $L^\alpha V$ is indecomposable, hence $L^\alpha V=\Sigma^\alpha$ and so it is simple. The claim follows. □

[2] A fundamental domain is a complete irredundant set of orbit representatives [Bou02, IV.3.3].

The fact that $\bigoplus_{\alpha\in B_{l,m-l}} \bigwedge^{\alpha'} \mathcal{Q}$ is a tilting object, together with Theorem 5.6 and the previous lemma, immediately yields the following.

Corollary 7.8 (Kaneda [Kan08]). *The Kapranov strong exceptional collection $(L^\alpha\mathcal{Q})_{\alpha\in B_{l,m-l}}$ remains strong exceptional as long as $p \geqslant m-1$. In particular, $\mathcal{K} = \bigoplus_{\alpha\in B_{l,m-l}} L^\alpha\mathcal{Q}$ remains tilting for such p.*

8. Relation with quasi-hereditary algebras

We quickly remind the reader of the module-theoretic description of quasi-hereditary algebras *à la* Dlab and Ringel [DR92]; see also, for example, [Erd94, HP11]. Let A be a finite-dimensional K-algebra and let $(S(\lambda))_{\lambda\in\Lambda}$ be a complete set of the simple modules, with projective covers $P(\lambda) \twoheadrightarrow S(\lambda)$ and injective hulls $S(\lambda) \hookrightarrow Q(\lambda)$.

Fix a total ordering $\prec$ on Λ. Define the *standard module* $\Delta(\lambda)$ to be the largest quotient of $P(\lambda)$ having composition factors of the form $S(\mu)$ with $\mu \preceq \lambda$. Equivalently [DR92, Lemma 1.1], $\Delta(\lambda)$ is the quotient of $P(\lambda)$ by the maximal submodule generated by any direct sum of the form $\bigoplus_{\mu\succ\lambda} P(\mu)$. Similarly, the *costandard module* $\nabla(\lambda)$ is the largest submodule of $Q(\lambda)$ having composition factors $S(\mu)$ with $\mu \preceq \lambda$. In particular, $S(\lambda)$ is the top of $\Delta(\lambda)$ and the socle of $\nabla(\lambda)$. Set $\Delta = (\Delta(\lambda))_{\lambda\in\Lambda}$ and $\nabla = (\nabla(\lambda))_{\lambda\in\Lambda}$.

We assume that each $\Delta(\lambda)$ (equivalently each $\nabla(\lambda)$) is Schurian, i.e. the endomorphism ring is a division ring.

For an arbitrary collection Θ of A-modules, denote by $\mathcal{F}(\Theta)$ the class of Θ-filtered modules, i.e. A-modules M having a filtration $M = M_0 \supset M_1 \supset \cdots \supset M_t = 0$ with successive quotients M_i/M_{i-1} in the collection Θ.

Definition 8.1 ([DR92, Theorem 1]; see also [Don81, Sco87, CPS88]). The algebra A (with the fixed order $\prec$) is called *quasi-hereditary* if the following equivalent conditions hold:

(i) ${}_AA \in \mathcal{F}(\Delta)$;
(ii) $\mathcal{F}(\Delta) = \{X \mid \operatorname{Ext}^1_A(X, \nabla(\lambda)) = 0\}$;
(iii) $\mathcal{F}(\Delta) = \{X \mid \operatorname{Ext}^i_A(X, \nabla(\lambda)) = 0 \text{ for all } i \geqslant 1\}$;
(iv) $\operatorname{Ext}^2_A(\Delta, \nabla) = 0$.

For a quasi-hereditary algebra, the standard and costandard modules determine each other in $D^b_f(A)$ (the bounded derived category with finite cohomology) by

$$\operatorname{RHom}_A(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu} \cdot K. \tag{8.1}$$

For the proof of Theorem 8.3 below we also remind the reader of the notion of 'standardization' [DR92, § 3], see also the 'universal extensions' of [HP11], in the special case of interest. An indexed collection $\Theta = (\Theta(\lambda))_{\lambda\in\Lambda}$ of objects in a K-linear abelian category $\mathcal{C}$ is *standardizable* provided:

(i) $\operatorname{Hom}_{\mathcal{C}}(\Theta(\lambda),\ \Theta(\mu))$ and $\operatorname{Ext}^1_{\mathcal{C}}(\Theta(\lambda)\ \Theta(\mu))$ are finite-dimensional for all λ, μ; and
(ii) the quiver with vertex set Λ and an arrow $\lambda \longrightarrow \mu$ if either there is a non-trivial non-isomorphism $\Theta(\lambda) \longrightarrow \Theta(\mu)$ or $\operatorname{Ext}^1_{\mathcal{C}}(\Theta(\lambda),\ \Theta(\mu)) \neq 0$ has no oriented cycles.

In particular, note that if Θ is an exceptional collection then Θ is standardizable.

The next result is essentially contained in the proof of [DR92, Theorem 2] for modules over a finite-dimensional algebra; see also [HP11, Theorem 5.1] for a statement in the geometric context.

Theorem 8.2. *Let $\Theta = (\Theta(\lambda))_{\lambda\in\Lambda}$ be a standardizable collection in an abelian category $\mathcal{C}$. Then there exists a projective generator $P \in \mathcal{F}(\Theta)$ such that $A' = \operatorname{End}_{\mathcal{C}}(P)$ is quasi-hereditary with standard modules $\operatorname{Hom}_{\mathcal{C}}(P, \Theta(\lambda))$.*

As in the Introduction, put

$$\mathcal{T} = \bigoplus_{\alpha\in B_{l,m-l}} \bigwedge^{\alpha'} \mathcal{Q}$$

and $A = \operatorname{End}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T})$. Denote by A° the opposite algebra. For $\alpha \in B_{l,m-l}$, consider the following complexes of right A-modules:

$$\Delta(\alpha) = \operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, L^{\alpha}\mathcal{Q}), \tag{8.2}$$
$$\nabla(\alpha) = \operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, L^{\alpha'}\mathcal{R}[|\alpha|]), \tag{8.3}$$
$$P(\alpha) = \operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, \mathcal{L}_{\mathbb{G}}(M^{\alpha})), \tag{8.4}$$
$$S(\alpha) = \operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, \mathcal{L}_{\mathbb{G}}(\Sigma'^{\alpha'})[|\alpha|]), \tag{8.5}$$

where, as before, M^{α} is the indecomposable tilting $\mathrm{GL}(l)$-representation with highest weight α.

Theorem 8.3. *The complexes $\Delta(\alpha)$, $\nabla(\alpha)$, $P(\alpha)$, $S(\alpha)$ are concentrated in degree zero, and the $S(\alpha)$ are the simple right A-modules with the $P(\alpha)$ their projective covers.*

As in § 7, let $\prec$ be an arbitrary total ordering on $B_{l,m-l}$ such that if $|\alpha| < |\beta|$, or if $|\alpha| = |\beta|$ and $\alpha > \beta$ in the natural partial order on partitions, then $\alpha \prec \beta$. Then A is quasi-hereditary with respect to this ordering. The standard and costandard modules having $S(\alpha)$ respectively as top and socle are $\Delta(\alpha)$ and $\nabla(\alpha)$.

Proof. Set $\Theta = (L^{\alpha}\mathcal{Q})_{\alpha\in B_{l,m-l}}$. Then Θ is an exceptional collection by Theorem 7.5, so, in particular, is standardizable. Let $P \in \mathcal{F}(\Theta)$ be the projective generator guaranteed by Theorem 8.2, so that $A' = \operatorname{End}_{\mathcal{O}_{\mathbb{G}}}(P)$ is quasi-hereditary with standard modules $\operatorname{Hom}_{\mathcal{O}_{\mathbb{G}}}(P, L^{\alpha}\mathcal{Q})$.

On the other hand, Proposition 1.4 implies that $\mathcal{T}$ is a projective generator for $\mathcal{F}(\Theta)$ as well. This easily yields that A and A' are Morita equivalent and that the objects $\operatorname{Hom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, L^{\alpha}\mathcal{Q})$ are the standard objects. By Proposition 1.4, we may replace Hom by RHom.

Since the costandard modules $\nabla(\beta)$ are characterized by $\operatorname{RHom}_{A^\circ}(\Delta(\alpha),\ \nabla(\beta)) = \delta_{\alpha,\beta}\cdot K$, we deduce from (7.3) that they are indeed given by the formula (8.3).

By [Don93, Lemma (3.4)] the indecomposable summands of $\bigwedge^{\alpha'} V$ are precisely the tilting representations M^{β}, and M^{α} occurs as a direct summand with multiplicity one in $\bigwedge^{\alpha'} V$. It follows that the $P(\alpha)$ are indeed the indecomposable projectives.

To show that the $S(\alpha)$ are the corresponding simple A-modules we have to prove

$$\operatorname{RHom}_{A^\circ}(P(\alpha), S(\beta)) = \delta_{\alpha,\beta}\cdot K.$$

We compute

$$\operatorname{RHom}_{A^\circ}(P(\alpha), S(\beta)) = \operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{L}_{\mathbb{G}}(M^{\alpha}), \mathcal{L}_{\mathbb{G}}(\Sigma'^{\beta'})[|\beta|]).$$

It follows from Theorem 5.10 combined with (4.1) that if $|\alpha| \neq |\beta|$ then there is nothing to prove. So assume $|\alpha| = |\beta| = d$. Then we have

$$\begin{aligned}\operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{L}_{\mathbb{G}}(M^{\alpha}), \mathcal{L}_{\mathbb{G}}(\Sigma'^{\beta'})[|\beta|]) &= \operatorname{RHom}_{\mathcal{O}_{\mathbb{G}}}(\gamma_d(\Phi_d(M^{\alpha})), \Phi'_d(\Sigma'^{\beta'})[|\beta|])\\ &= \operatorname{RHom}_{G_2}(\delta_d(M^{\alpha}), \Sigma'^{\beta'})\\ &= \operatorname{RHom}_{G_2}(P'^{\alpha'}, \Sigma'^{\beta'})\\ &= \delta_{\alpha,\beta}\cdot K,\end{aligned}$$

where the first equality is Lemma 4.3 and the third is Proposition 6.1; in the second, δ_d is as introduced in § 6.

It remains to prove that $S(\alpha)$ is the top of $\Delta(\alpha)$ and the socle of $\nabla(\alpha)$. Since for quasi-hereditary algebras the top of $\Delta(\alpha)$ coincides with the socle of $\nabla(\alpha)$ it is sufficient to prove only the first of these statements. There is a surjective map $M^\alpha \longrightarrow L^\alpha V$ whose kernel is an extension of $L^\beta V$ with $\beta < \alpha$ [Jan03, § E.4]. Apply $\operatorname{Hom}_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T}, \mathcal{L}_{\mathbb{G}}(-))$ to obtain a map $P(\alpha) \longrightarrow \Delta(\alpha)$, which is surjective by Proposition 1.4. This finishes the proof. $\square$

Example 8.4. We compute the quiver and relations of the quasi-hereditary algebra A in the first non-trivial example, $(m, l) = (4, 2)$. We live inside the 2×2 box $B_{2,2}$, so the vertices of the quiver, equivalently the summands of the tilting bundle $\mathcal{T}$, are labeled

$$\mathcal{O},\ \mathcal{Q},\ \textstyle\bigwedge^2\mathcal{Q},\ \mathcal{Q}\otimes\mathcal{Q},\ \bigwedge^2\mathcal{Q}\otimes\mathcal{Q},\ (\bigwedge^2\mathcal{Q})^{\otimes 2}.$$

The quiver has the following form:

$$\begin{tikzcd}
 & & \bigwedge^2\mathcal{Q} \arrow[dd, shift left, "a"] & & \\
\mathcal{O} \arrow[r, "s_\lambda"] & \mathcal{Q} \arrow[dr, "\alpha_\lambda"'] & & \bigwedge^2\mathcal{Q}\otimes\mathcal{Q} \arrow[r, "t_\lambda"] & (\bigwedge^2\mathcal{Q})^{\otimes 2} \\
 & & \mathcal{Q}\otimes\mathcal{Q} \arrow[uu, shift left, "p"] \arrow[ur, "\beta_\lambda"'] & &
\end{tikzcd}$$

The labels stand for natural maps between these bundles, some of which depend on a global section $\lambda \in F^\vee$:

- $p : \mathcal{Q}\otimes\mathcal{Q} \longrightarrow \bigwedge^2\mathcal{Q}$ the natural surjection and $a : \bigwedge^2\mathcal{Q} \longrightarrow \mathcal{Q}\otimes\mathcal{Q}$ the anti-symmetrization;
- $s_\lambda : \mathcal{O} \longrightarrow \mathcal{Q}$ with $1 \mapsto \lambda$;
- $\alpha_\lambda : \mathcal{Q} \longrightarrow \mathcal{Q}\otimes\mathcal{Q}$ with $x \mapsto \lambda\otimes x$;
- $\beta_\lambda : \mathcal{Q}\otimes\mathcal{Q} \longrightarrow \bigwedge^2\mathcal{Q}\otimes\mathcal{Q}$ with $x\otimes y \mapsto \lambda\wedge y\otimes x$; and
- $t_\lambda : \bigwedge^2\mathcal{Q}\otimes\mathcal{Q} \longrightarrow (\bigwedge^2\mathcal{Q})^{\otimes 2}$ with $x\wedge y\otimes z \mapsto x\wedge y\otimes\lambda\wedge z$.

These maps generate all the arrows in the quiver. For example, the obvious complementary map $\alpha'_\lambda : \mathcal{Q} \longrightarrow \mathcal{Q}\otimes\mathcal{Q}$ defined by $\alpha'_\lambda(x) = x\otimes\lambda$ can be obtained as $\alpha_\lambda(1 - ap)$. The relations are most compactly written in terms of the pseudo-idempotent $e \overset{\text{def}}{=} ap$ satisfying $e^2 = 2e$, and the 'swap' $1 - e$ which sends $x\otimes y$ to $y\otimes x$. We have:

- $pa = 2\operatorname{Id}_{\bigwedge^2\mathcal{Q}}$;
- $(1-e)\alpha_\lambda s_\mu = \alpha_\mu s_\lambda$;
- $\beta_\lambda(1-e)\alpha_\mu = \beta_\lambda\alpha_\mu - \beta_\mu\alpha_\lambda$;
- $t_\lambda\beta_\mu(1-e) = t_\mu\beta_\lambda$.

Indeed, since $\mathcal{Q}$ has rank two, we have $\bigwedge^2\mathcal{Q}\otimes\mathcal{Q} = L^{[2,1]}\mathcal{Q}$, in which the exchange relation

$$x\wedge y\otimes z + y\wedge z\otimes x + z\wedge x\otimes y = 0$$

holds.

We observe that in this picture, each vertical 'slice' is equivalent to the derived category of a generalized Schur algebra. For example, in the middle we recognize the quiver for the Schur algebra $S(2,2)$ in characteristic two [Erd93, 3.1.1, 5.4].

In characteristic different from two, the idempotent $\frac{1}{2}e$ gives $\mathcal{Q} \otimes \mathcal{Q} \cong \bigwedge^2 \mathcal{Q} \oplus \operatorname{Sym}^2 \mathcal{Q}$, and the algebra becomes Morita equivalent to the path algebra of the equivariant quiver

$$\begin{tikzcd}
 & & \bigwedge^2 \mathcal{Q} \arrow[dr, "F^\vee"] \arrow[rr, dashed, bend left, "D^2 F^\vee"] & & \\
\mathcal{O} \arrow[r, "F^\vee"] \arrow[urr, dashed, bend left, "D^2 F^\vee"] \arrow[drr, dashed, bend right, "\bigwedge^2 F^\vee"'] & \mathcal{Q} \arrow[ur, "F^\vee"] \arrow[dr, "F^\vee"'] \arrow[rr, dashed, "F^\vee \otimes F^\vee"] & & \bigwedge^2 \mathcal{Q} \otimes \mathcal{Q} \arrow[r, "F^\vee"] & (\bigwedge^2 \mathcal{Q})^{\otimes 2} \\
 & & \operatorname{Sym}^2 \mathcal{Q} \arrow[ur, "F^\vee"'] \arrow[urr, dashed, bend right, "\bigwedge^2 F^\vee"'] & &
\end{tikzcd}$$

with relations (indicated by dashed arrows)

- $\mathcal{O} \longrightarrow \bigwedge^2 \mathcal{Q}$ given by $D^2 F^\vee$;
- $\mathcal{O} \longrightarrow \operatorname{Sym}^2 \mathcal{Q}$ given by $\bigwedge^2 F^\vee$;
- $\mathcal{Q} \longrightarrow \bigwedge^2 \mathcal{Q} \otimes \mathcal{Q}$ given by $F^\vee \otimes F^\vee$;
- $\bigwedge^2 \mathcal{Q} \longrightarrow (\bigwedge^2 \mathcal{Q})^{\otimes 2}$ given by $D^2 F^\vee$; and
- $\operatorname{Sym}^2 \mathcal{Q} \longrightarrow (\bigwedge^2 \mathcal{Q})^{\otimes 2}$ given by $\bigwedge^2 F^\vee$.

Most of these are straightforward to verify. The relations across the central diamond, however, are not the obvious commutativity ones [Hil98, SW11]. To compute those relations, give names to the maps:

$$\begin{tikzcd}
 & \bigwedge^2 \mathcal{Q} \arrow[dr, "b_\lambda"] & \\
\mathcal{Q} \arrow[ur, "a_\lambda"] \arrow[dr, "c_\lambda"'] & & \bigwedge^2 \mathcal{Q} \otimes \mathcal{Q} \\
 & \operatorname{Sym}^2 \mathcal{Q} \arrow[ur, "d_\lambda"'] &
\end{tikzcd}$$

with

- $a_\lambda(x) = \lambda \wedge x$;
- $b_\lambda(x \wedge y) = x \wedge y \otimes \lambda$;
- $c_\lambda(x) = \lambda x$; and
- $d_\lambda(xy) = \lambda \wedge x \otimes y + \lambda \wedge y \otimes x$.

Then we find, for $\lambda, \mu \in F^\vee$,

$$d_\mu c_\lambda = 2 b_\lambda a_\mu - b_\mu a_\lambda.$$

It follows that the defining relations are

$$\begin{aligned} d_\lambda c_\mu + d_\mu c_\lambda &= b_\lambda a_\mu + b_\mu a_\lambda, \\ d_\mu c_\lambda - d_\lambda c_\mu &= 3(b_\lambda a_\mu - b_\mu a_\lambda). \end{aligned}$$

Acknowledgements

The authors wish to thank Vincent Franjou, Catharina Stroppel and Antoine Touzé for help with references. We are also grateful to the referees for carefully reading the manuscript and suggesting many improvements. After this paper was posted to the arXiv, we learned that Alexander Efimov has obtained results similar to Theorems 1.6 and 1.7 by different methods.

References

Bof88 G. Boffi, *The universal form of the Littlewood–Richardson rule*, Adv. Math. **68** (1988), 40–63; MR 931171.

Bon89 A. I. Bondal, *Representations of associative algebras and coherent sheaves*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), 25–44; MR 992977.

BK89 A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), 1183–1205, 1337; MR 1039961.

BVdB03 A. I. Bondal and M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), 1–36, 258; MR 1996800.

Bou02 N. Bourbaki, *Lie groups and Lie algebras*, Elements of Mathematics (Berlin) (Springer, Berlin, 2002), ch. 4–6. Translated from the 1968 French original by Andrew Pressley; MR 1890629.

BLVdB13 R.-O. Buchweitz, G. J. Leuschke and M. Van den Bergh, *Non-commutative desingularization of determinantal varieties, II: Arbitrary minors*, Preprint (2013), arXiv:1106.1833.

CPS83 E. T. Cline, B. Parshall and L. Scott, *A Mackey imprimitivity theory for algebraic groups*, Math. Z. **182** (1983), 447–471; MR 701363.

CPS88 E. T. Cline, B. Parshall and L. Scott, *Finite-dimensional algebras and highest weight categories*, J. Reine Angew. Math. **391** (1988), 85–99; MR 961165.

DR92 V. Dlab and C. M. Ringel, *The module theoretical approach to quasi-hereditary algebras*, in *Representations of algebras and related topics (Kyoto, 1990)*, London Mathematical Society Lecture Note Series, vol. 168 (Cambridge University Press, Cambridge, 1992), 200–224; MR 1211481.

Don81 S. Donkin, *A filtration for rational modules*, Math. Z. **177** (1981), 1–8; MR 611465.

Don93 S. Donkin, *On tilting modules for algebraic groups*, Math. Z. **212** (1993), 39–60; MR 1200163.

DRS74 P. Doubilet, G.-C. Rota and J. Stein, *On the foundations of combinatorial theory. IX. Combinatorial methods in invariant theory*, Stud. Appl. Math. **53** (1974), 185–216; MR 0498650.

Erd93 K. Erdmann, *Schur algebras of finite type*, Quart. J. Math. Oxford Ser. (2) **44** (1993), 17–41; MR 1206201.

Erd94 K. Erdmann, *Symmetric groups and quasi-hereditary algebras*, in *Finite-dimensional algebras and related topics (Ottawa, ON, 1992)*, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences vol. 424 (Kluwer Academic Publishing, Dordrecht, 1994), 123–161; MR 1308984.

Ful97 W. Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35 (Cambridge University Press, Cambridge, 1997); MR 1464693.

Hil98 L. Hille, *Homogeneous vector bundles and Koszul algebras*, Math. Nachr. **191** (1998), 189–195; MR 1621314.

HP11 L. Hille and M. Perling, *Tilting bundles on rational surfaces and quasi-hereditary algebras*, Preprint (2011), arXiv:1110.5843.

Jan03 J. C. Jantzen, *Representations of algebraic groups*, Mathematical Surveys and Monographs, vol. 107, second edition (American Mathematical Society, Providence, RI, 2003); MR 2015057.

Kan08 M. Kaneda, *Kapranov's tilting sheaf on the Grassmannian in positive characteristic*, Algebr. Represent. Theory **11** (2008), 347–354; MR 2417509.

Kap88 M. M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. **92** (1988), 479–508; MR 939472.

Kem76 G. R. Kempf, *Linear systems on homogeneous spaces*, Ann. of Math. (2) **103** (1976), 557–591; MR 0409474.

Kuz09 A. Kuznetsov, *Hochschild homology and semiorthogonal decompositions*, Preprint (2009), arXiv:0904.4330.

LSW89 M. Levine, V. Srinivas and J. Weyman, *K-theory of twisted Grassmannians*, K-Theory **3** (1989), 99–121; MR 1029954.

SW11 S. V. Sam and J. Weyman, *Pieri resolutions for classical groups*, J. Algebra **329** (2011), 222–259. See also the revised and augmented version at arXiv:0907.4505v5; MR 2769324.

Sco87 L. L. Scott, *Simulating algebraic geometry with algebra. I*, in *The algebraic theory of derived categories, The Arcata conference on representations of finite groups (Arcata, California, 1986), Proceedings of Symposia in Pure Mathematics,* vol. 47 (American Mathematical Society, Providence, RI, 1987), 271–281; MR 933417.

Wey03 J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics vol. 149 (Cambridge University Press, Cambridge, 2003); MR 1988690.

Ragnar-Olaf Buchweitz ragnar@utsc.utoronto.ca
Department of Computer and Mathematical Sciences, University of Toronto Scarborough, Toronto, Ontario M1C 1A4, Canada

Graham J. Leuschke gjleusch@math.syr.edu
Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA

Michel Van den Bergh michel.vandenbergh@uhasselt.be
Departement WNI, Universiteit Hasselt, 3590 Diepenbeek, Belgium