

ERRATUM TO “ENDOMORPHISM RINGS OF FINITE GLOBAL DIMENSION”

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This note corrects three points in [Leu07]. The corrections in Sections 1 and 2 were recorded briefly in the survey [Leu12] (Theorem 17.2 and its footnote, and p. 338); they are collected here, with proofs, for readers of [Leu07]. Numbering refers to the published version of [Leu07]. Throughout, $(R, \mathfrak{m}, k)$ is a d -dimensional CM local ring of finite CM type, M is a representation generator for R , and $A = \text{End}_R(M)^{\text{op}}$, as in Theorem 6 of [Leu07].

1. THE EQUALITY IN THEOREM 6

Theorem 6 asserts $\text{gldim } A \leq \max\{2, d\}$, with equality if $d \geq 2$. The proof of the inequality is correct. The proof of the equality is not: it asserts that a projective resolution of a simple A -module is “in particular an exact sequence of MCM R -modules.” The projective A -modules are the $\text{Hom}_R(M, M')$ with $M' \in \text{add } M$, and these need not be MCM R -modules; in Example 11 of [Leu07], $\text{Hom}_R(M, M)$ has depth 2 over a ring of dimension 3.

Christopher Quarles pointed this out in May 2005, and Igor Burban in 2009. Quarles observed further that the argument cannot work for the simple A -modules S_Z attached to the nonfree indecomposable MCM modules Z : when AR sequences exist (for example, when R is complete) and $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is the AR sequence ending in Z ,

$$0 \rightarrow \text{Hom}_R(M, X) \rightarrow \text{Hom}_R(M, Y) \rightarrow \text{Hom}_R(M, Z) \rightarrow S_Z \rightarrow 0$$

is a projective resolution, so $\text{pd}_A S_Z \leq 2$ in every dimension.

The equality holds nonetheless, as noted in [Leu12, Theorem 17.2]. Suppose $d \geq 2$. Since R is CM with an isolated singularity, it satisfies Serre’s conditions (R_1) and (S_2) , so it is a normal domain, and A is torsion-free as an R -module because M is. By [Ram69, Corollary 3.2], $\text{injdim}_A A \geq d$. Since $\text{injdim}_A A \leq \text{gldim } A$ for any ring, $\text{gldim } A \geq d$. With the upper bound of Theorem 6, $\text{gldim } A = \max\{2, d\}$, so the statement of Theorem 6 stands as published. So do the assertions in Examples 11 and 12 that A has global dimension 3, and the equality $\text{gldim } A = 2$ used for Definition 10 in dimension two.

When AR sequences exist, Quarles’s resolutions show which simple module carries the global dimension: $\text{pd}_A S_Z \leq 2$ for every nonfree Z , so if $d \geq 3$ the simple module $S_R = \text{Hom}_R(M, R)/\text{Hom}_R(M, \mathfrak{m})$ attached to the free summand has projective dimension d .

Remark 1.1. The inequality $\text{pd}_A S_R \geq d$ can also be seen directly, without AR sequences, and for any MCM R -module M with $R \in \text{add } M$. Let $e \in A$ be the idempotent $M \twoheadrightarrow R \hookrightarrow M$. The functor $X \mapsto eX$ from A -modules to R -modules is exact and carries $\text{Hom}_R(M, L)$ to $\text{Hom}_R(R, L) = L$ for every R -module L ; in particular $eS_R = k$. By projectivization [ARS97, II.2], a projective resolution of S_R has the form $\text{Hom}_R(M, M_\bullet)$ with each $M_i \in \text{add } M$, and applying e turns it into a resolution $M_\bullet \rightarrow k \rightarrow 0$ by MCM R -modules. Since $\text{depth } k = 0$, the depth lemma shows that its length is at least d .

2. NON-COMMUTATIVE CREPANT RESOLUTIONS IN DIMENSION THREE

The following statements in [Leu07] are false in dimension three:

Date: September 29, 2026.

- (a) the last sentence of the abstract, that for R Gorenstein the construction gives a non-commutative crepant resolution;
- (b) on p. 333, that if R is Gorenstein of dimension 2 or 3 and has finite CM type, Theorem 6 produces a non-commutative crepant resolution of R ;
- (c) on p. 340, that Theorem 6 implies that a Gorenstein local ring of finite CM type containing a field and of dimension two or three has a non-commutative crepant resolution, and that for each simple singularity the endomorphism ring of a representation generator is an MCM module.

Igor Burban pointed this out in June 2009, and Hailong Dao in October 2009. Dao's example: for $R = \mathbb{C}[[x, y, u, v]]/(xu - yv)$ the nonfree indecomposable MCM modules are $I = (x, y)$ and $I^* \cong (x, v)$, and $\text{Hom}_R(I^*, I)$ has depth 2.

In [Leu12, p. 338] the failure is deduced from the projective dimensions of the simple A -modules: $\text{pd}_A S_Z = 2$ for Z nonfree while $\text{pd}_A S_R = d$, so A is not homologically homogeneous when $d \geq 3$. The failure of the MCM property is uniform in dimension at least three.

Proposition 2.1. *Let $(R, \mathfrak{m})$ be a CM local ring of finite CM type and dimension $d \geq 3$, not regular, and let M be a representation generator. Then $\text{depth}_R \text{End}_R(M) = 2$. In particular $\text{End}_R(M)$ is not an MCM R -module.*

Proof. For MCM modules X and Y , a free presentation $F_1 \rightarrow F_0 \rightarrow X \rightarrow 0$ gives an exact sequence $0 \rightarrow \text{Hom}_R(X, Y) \rightarrow \text{Hom}_R(F_0, Y) \rightarrow \text{Hom}_R(F_1, Y)$, so $\text{depth} \text{Hom}_R(X, Y) \geq 2$ by the depth lemma. Hence $\text{depth} \text{End}_R(M) \geq 2$.

Since R is not regular there is a nonfree indecomposable MCM module Z . Let $0 \rightarrow Y \rightarrow F \rightarrow Z \rightarrow 0$ be a minimal free cover, so that $Y = \text{syzy}_1^R Z$ is an MCM module in $\text{add } M$. The sequence does not split, so $E := \text{Ext}_R^1(Z, Y) \neq 0$, and E has finite length since R has an isolated singularity. Applying $\text{Hom}_R(-, Y)$ gives

$$0 \rightarrow \text{Hom}_R(Z, Y) \rightarrow \text{Hom}_R(F, Y) \rightarrow \text{Hom}_R(Y, Y) \rightarrow E \rightarrow 0.$$

Let C be the image of $\text{Hom}_R(F, Y) \rightarrow \text{Hom}_R(Y, Y)$. From $0 \rightarrow C \rightarrow \text{Hom}_R(Y, Y) \rightarrow E \rightarrow 0$, with $\text{depth} \text{Hom}_R(Y, Y) \geq 2$ and $\text{depth } E = 0$, we get $\text{depth } C = 1$. From $0 \rightarrow \text{Hom}_R(Z, Y) \rightarrow \text{Hom}_R(F, Y) \rightarrow C \rightarrow 0$, with $\text{depth} \text{Hom}_R(F, Y) = d \geq 3$, we get $\text{depth} \text{Hom}_R(Z, Y) = 2$. Since $\text{Hom}_R(Z, Y)$ is a direct summand of $\text{End}_R(M)$, the depth of $\text{End}_R(M)$ is at most 2. $\square$

When R is a finitely generated free module over a complete regular local ring, Proposition 2.1 is also a consequence of Iyama's criterion [Iya07, 2.5.1]: for MCM modules X and Y over an isolated singularity, $\text{depth} \text{Hom}_R(X, Y) \geq 3$ if and only if $\text{Ext}_R^1(X, Y) = 0$. See also [Leu12, Lemma 14.2].

So in dimension three the endomorphism ring of a representation generator is never a non-commutative crepant resolution. Statements (a)–(c) are correct in dimension two, where $\text{End}_R(M)$ is reflexive, hence MCM, for every MCM module M over a normal domain. Whether a three-dimensional simple singularity has a non-commutative crepant resolution by some other module depends on the singularity: the A_2 singularity $x^2 + y^3 + uv$ has none [Van04, Example A.1], and the question is settled for all three-dimensional simple singularities in [BIKR08].

On p. 340 it is also said that Theorem 6 implies that graded Gorenstein local rings of finite CM type containing $\mathbb{C}$ have rational singularities in dimensions 2 and 3. In dimension three this does not follow from Theorem 6, by Proposition 2.1; the conclusion itself holds by the classification, as noted there.

Proposition 2.1 also accounts for the failure of the MCM property in Examples 11 and 12, and is consistent with Remark 9.

3. PROPOSITION 8

Definition 7 requires R to be a finitely generated free module over a complete regular local ring T , and the proof of Proposition 8 obtains T from Cohen's structure theorem. Not every complete

CM local ring of finite CM type is free over a complete regular local subring: $R = \mathbb{Z}/p^2\mathbb{Z}$ is Artinian with two indecomposable modules, and contains no field. For such R , Definition 7 does not define $\text{repdim} R$.

A complete CM local ring R with residue characteristic p is a finitely generated free module over some complete regular local ring if and only if R is equicharacteristic or p is a nonzerodivisor on R . If p is a nonzerodivisor, extend it to a system of parameters $p, x_2, \dots, x_d$ and take $T = W[[x_2, \dots, x_d]]$ with W a coefficient ring. Conversely, a regular local ring T is a domain, so $T \subseteq R$ forces $\text{char} R \in \{0, p\}$, and in characteristic 0 the element p is a nonzerodivisor on T , hence on the free T -module R . With this hypothesis added, Proposition 8 and its proof are correct.

Acknowledgements. I thank Christopher Quarles, Igor Burban, and Hailong Dao for pointing out the errors corrected in [Sections 1](#) and [2](#).

The LLM Claude Opus 5 (Anthropic) drafted an initial version of this erratum from the author's notes. The model also contributed the alternative proof of Theorem 6 given in [Remark 1.1](#).

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