A NEW PROOF OF CHARACTERISTIC-FREE KNÖRRER PERIODICITY

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Dedicated to the memory of Ragnar-Olaf Buchweitz

ABSTRACT. Let $R = S/(f)$ be a complete local hypersurface ring and let $A = S[\![u,v]\!]/(f+uv)$. Pfister and Popescu showed that the stable categories of maximal Cohen-Macaulay modules over R and over A are equivalent, with no assumption on the characteristic. We give a short proof. If (Φ,Ψ) is a matrix factorization of h and ∂ is a derivation, then $(\partial\Psi,\partial\Phi)$ is a null-homotopy for multiplication by ∂h. Since u and v are partial derivatives of $f+uv$, they act as zero on the stable category of A, so the Koszul complex on u and v splits in the singularity category, and this gives density. For the double branched cover $S[\![z]\!]/(f+z^2)$ the same argument yields $2z$: it recovers Knörrer's lemma up to free summands on the double branched cover when 2 is a unit, and we show by an example that this lemma, and periodicity for the iterated double branched cover $S[\![z,w]\!]/(f+z^2+w^2)$, fail in characteristic two. We also compute the second syzygy over A of a maximal Cohen-Macaulay R-module by an explicit resolution, and show that the change of basis used there cannot be given by a universal matrix formula.

1. INTRODUCTION

Let $(S,\mathfrak{n},k)$ be a complete regular local ring, $f \in \mathfrak{n}^2$ a non-zero element, and $R = S/(f)$. Knörrer periodicity [Knö87, Thm. 3.1] states that when k is algebraically closed of characteristic different from 2, the stable categories of maximal Cohen-Macaulay (MCM) modules over R and over $R^{\sharp\sharp} = S[\![z,w]\!]/(f+z^2+w^2)$ are equivalent. The proof passes through the double branched cover $R^{\sharp} = S[\![z]\!]/(f+z^2)$, and that is where the characteristic enters: one step diagonalizes a matrix by a change of basis of determinant 2. The change of variables $u = z+iw$, $v = z-iw$ identifies $R^{\sharp\sharp}$ with the hyperbolic extension $A = S[\![u,v]\!]/(f+uv)$ as long as 2 is a unit, but Knörrer's proof does not apply to give an equivalence between $\underline{\mathrm{MCM}}\,R$ and $\underline{\mathrm{MCM}}\,A$ if 2 is not a unit. Indeed, in characteristic 2 the rings A and $R^{\sharp\sharp}$ differ, and periodicity for $R^{\sharp\sharp}$ fails (see Proposition 5.4).

Pfister and Popescu proved that $\underline{\mathrm{MCM}}\,R \simeq \underline{\mathrm{MCM}}\,A$ with no hypothesis on S or f [PP96, Thm. 3.12]. This note gives a short proof of Pfister and Popescu's theorem which shows where the 2 comes from. If (Φ,Ψ) is a matrix factorization of h and ∂ is a derivation, differentiating $\Phi\Psi = h\cdot 1$ shows that $(\partial\Psi,\partial\Phi)$ is a null-homotopy for multiplication by ∂h, so ∂h acts as zero on the stable category. For $h = f+uv$ we have $\partial_v h = u$ and $\partial_u h = v$; so u and v act as zero, the Koszul complex on u,v splits in the singularity category of A, and every MCM A-module is a direct summand, in the singularity category, of a MCM R-module regarded as A-module by restriction of scalars. For $h = f+z^2$ the derivative is $2z$. When 2 is a unit the same argument proves Knörrer's lemma on $R^{\sharp}$ (Proposition 5.2); when $2=0$ it proves nothing, and the lemma is false (Proposition 5.4).

Section 2 fixes the setting and states the theorem, with consequences for Cohen-Macaulay representation type. Section 3 proves it: full faithfulness is due to Pfister-Popescu and to Solberg [Sol89], and density (Proposition 3.6) is the argument above. Section 4 computes $\mathrm{syz}^A_2 M$ for a MCM R-module M by an explicit resolution, in which the derivatives of (Φ,Ψ) appear as a change of basis, and shows that no universal polynomial formula can replace them. Section 5 treats the double branched cover:

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the Jacobian proof of Knörrer's lemma when 2 is a unit, and the example $f = x^2$ in characteristic 2, where the hyperbolic extension shows periodicity and the double branched cover and its iterate do not.

Acknowledgements. This paper grew out of an exchange with the LLM Claude about whether the characteristic $\neq 2$ hypothesis in Knörrer periodicity is essential. The model also compiled an initial draft. The author is responsible for all statements, proofs, and errors. An earlier version claimed priority for the main result, before the author was alerted to the previous work of [PP96].

I was asked the initial question — Is there a direct characteristic-free proof of the equivalence $\mathrm{MF}(f) \simeq \mathrm{MF}(f+uv)$? — by Osamu Iyama on November 12, 2017, during a very pleasant and productive visit to Nagoya. My friend and collaborator Ragnar-Olaf Buchweitz had just passed away the day before. I have returned to the question many times over the years since, always with the feeling that Ragnar would have had deeper insight into the question than I was able to. I am pleased that this proof uses so many of Ragnar's favorite tools.

2. The hyperbolic extension

2.1. Throughout, $(S, \mathfrak{n}, k)$ is a complete regular local ring of dimension $d+1 \geqslant 1$, f is a non-zero element of $\mathfrak{n}^2$, and

$$R = S/(f), \qquad B = S[\![u,v]\!], \qquad A = R^{\text{✠}} = B/(f+uv).$$

No hypothesis is placed on k; in particular S need not contain a field. We call A the *hyperbolic extension* of R.

A *matrix factorization* of f over S is a pair $(\varphi\colon G \longrightarrow F, \psi\colon F \longrightarrow G)$ of homomorphisms of free S-modules of the same finite rank with $\psi\varphi = f\cdot 1_G$ and $\varphi\psi = f\cdot 1_F$; then $\operatorname{cok}(\varphi,\psi) := \operatorname{cok}\varphi$ is a MCM R-module. The homotopy category $\underline{\mathrm{MF}}_S(f)$ of matrix factorizations is triangulated, with $\Sigma(\varphi,\psi) = (-\psi,-\varphi)$ and $\operatorname{cone}(\alpha,\beta) = \left(\begin{bmatrix}\varphi' & \alpha\\ 0 & -\psi\end{bmatrix}, \begin{bmatrix}\psi' & \beta\\ 0 & -\varphi\end{bmatrix}\right)$, and cok induces equivalences

$$\underline{\mathrm{MF}}_S(f) \simeq \underline{\mathrm{MCM}}\, R \simeq \mathrm{D_{sg}}(R)$$

with the stable category of MCM R-modules and the singularity category $\mathrm{D_{sg}}(R) = \mathrm{D^b}(\operatorname{mod} R)/\operatorname{perf}(R)$ [Buc21, Thm. 4.4.1], [Eis80, §6]. The second equivalence holds for every Gorenstein ring. Under it, Σ^{-1} is syz_1 on stable MCM modules.

For Γ a complete local ring, $\underline{\mathrm{MCM}}\,\Gamma$ is a Krull-Schmidt category: idempotents split, and stable MCM Γ-modules are isomorphic if and only if they are isomorphic in $\underline{\mathrm{MCM}}\,\Gamma$ [LW12, Theorem 1.8], [Kra15, Corollary 4.4].

Definition 2.2 ([LW12, Definition 8.31])**.**

(1) For $M = \operatorname{cok}(\varphi\colon G \longrightarrow F, \psi\colon F \longrightarrow G)$ a MCM R-module, set

$$M^{\text{✠}} = \operatorname{cok}\left(\begin{bmatrix}\varphi & -v\,1_F\\ u\,1_G & \psi\end{bmatrix}, \begin{bmatrix}\psi & v\,1_G\\ -u\,1_F & \varphi\end{bmatrix}\right), \tag{2.1}$$

a MCM A-module. For a morphism (α,β) of matrix factorizations, set $(\alpha,\beta)^{\text{✠}} = (\operatorname{diag}(\alpha,\beta), \operatorname{diag}(\beta,\alpha))$.

(2) For N a MCM A-module, set $N^{\flat\flat} = N/(u,v)N$, a MCM R-module since u,v is a regular sequence on N. Define $(-)^{\flat\flat}$ on morphisms by $u = v = 0$.

We use the same notation for matrix factorizations. Up to isomorphism, $(-)^{\text{✠}}$ is Knörrer's functor H [Knö87, §3] and the functor of Pfister and Popescu (Remark 2.5), and Yoshino identifies it with tensoring by the factorization (u,v) of uv [Yos98, Remark 1.3].

Lemma 2.3. *Let M be a MCM R-module. Then*

(i) $\left(M^{\text{✠}}\right)^{\flat\flat} \cong M \oplus \mathrm{syz}_1^R M$*;*

(ii) $\left(\operatorname{syz}_1^R M\right)^{\bigstar} \cong \operatorname{syz}_1^A\left(M^{\bigstar}\right)$*; and*

(iii) $(-)^{\bigstar}$ *carries null-homotopies to null-homotopies and commutes with suspensions and cones, so is a triangulated functor* $\underline{\mathrm{MF}}_S(f) \longrightarrow \underline{\mathrm{MF}}_B(f+uv)$*, equivalently* $\underline{\mathrm{MCM}}\,R \longrightarrow \underline{\mathrm{MCM}}\,A$*.*

Proof. For (i), set $u = v = 0$ in (2.1). For (ii), conjugation by $\operatorname{diag}(1,-1)$ turns the matrix presenting $\left(\operatorname{syz}_1^R M\right)^{\bigstar}$ into that presenting $\operatorname{syz}_1^A(M^{\bigstar})$. For (iii), $(-)^{\bigstar}$ commutes with Σ on the nose, and with cone up to swapping the two middle blocks of basis vectors. □

Theorem 2.4 ([PP96, Theorem 3.12])**.** *The functor* $(-)^{\bigstar}\colon \underline{\mathrm{MCM}}\,R \longrightarrow \underline{\mathrm{MCM}}\,A$ *is an equivalence of categories.*

Remark 2.5. Pfister and Popescu work over a Noetherian Henselian local ring Λ with an ideal $\mathfrak{a}$ such that $f \notin \mathfrak{a}$ and $\Lambda/\mathfrak{a}$ is a discrete valuation ring. With $\overline{R} = \Lambda/(f,\mathfrak{a})$, they prove that $(-)^{\bigstar}$ is an equivalence between the stable categories of modules M over $\Lambda/(f)$ and $\Lambda[\![u,v]\!]/(f+uv)$ such that$\operatorname{Tor}_i(\overline{R},M) = 0$ for $i \geqslant 1$ [PP96, Definition 1.10, Theorem 3.12]. If we take $\Lambda = S$, and let $\mathfrak{a}$ be generated by $x_1,\dots,x_d \in \mathfrak{n}$ whose images form a system of parameters for R and are linearly independent in $\mathfrak{n}/\mathfrak{n}^2$, then $S/\mathfrak{a}$ is a DVR, and $f \notin \mathfrak{a}$ since $R/\mathfrak{a}R$ has finite length. As $\underline{x}$ is R-regular and $\underline{x},u,v$ is A-regular, the Tor condition says exactly that M is MCM. Their functor [PP96, 3.11], $(\varphi,\psi) \mapsto \left(\left[\begin{smallmatrix}\varphi & v\\ u & -\psi\end{smallmatrix}\right], \left[\begin{smallmatrix}\psi & v\\ u & -\varphi\end{smallmatrix}\right]\right)$, becomes $(\varphi,\psi)^{\bigstar}$ after changing the sign of the second block of basis vectors.

The following corollary is [LW12, Thm. 8.33] without the standing hypothesis there that k be algebraically closed of characteristic different from 2.

Corollary 2.6. *The functor* $(-)^{\bigstar}$ *induces a bijection between the isomorphism classes of indecomposable non-free MCM* R*-modules and those of indecomposable non-free MCM* A*-modules.*

Proof. Both $\underline{\mathrm{MCM}}(R)$ and $\underline{\mathrm{MCM}}(A)$ have the Krull-Remak-Schmidt uniqueness property for direct-sum decompositions, so the equivalence $(-)^{\bigstar}$ induces a bijection on the isomorphism classes of the indecomposable objects. □

The next corollary generalizes the main result of [Sol89], without assuming the existence of almost split sequences. Combined with Greuel-Kröning [GK90] it gives the "only if" direction of [LW12, Theorem 9.28].

Corollary 2.7. *The hypersurface defined by* f *has finite (resp., countable, resp., bounded) Cohen-Macaulay type if and only if that defined by* $f+uv$ *does.* □

Nothing here requires f to be reduced or to define an isolated singularity. For example, with $\operatorname{char} k = 2$ and $f = x^2y^2 \in k[\![x,y]\!]$, the ring A has a one-dimensional singular locus, and neither [Knö87] nor [Sol89] applies.

3. Proof of Theorem 2.4

The key computation holds for any matrix factorization and any derivation.

Lemma 3.1. *Let* T *be a commutative ring,* $h \in T$*, and* $(\Phi\colon \mathcal{G} \longrightarrow \mathcal{F}, \Psi\colon \mathcal{F} \longrightarrow \mathcal{G})$ *a matrix factorization of* h *over* T*. Let* ∂ *be a derivation of* T*, extended entrywise to matrices. Then*

$$(\partial\Phi)\Psi + \Phi(\partial\Psi) = (\partial h)\cdot 1_{\mathcal{F}} \qquad \text{and} \qquad (\partial\Psi)\Phi + \Psi(\partial\Phi) = (\partial h)\cdot 1_{\mathcal{G}}\,. \tag{3.1}$$

So $(\partial\Psi,\partial\Phi)$ *is a null-homotopy for multiplication by* ∂h*, and the Jacobian ideal* $\operatorname{jac}(h)$ *annihilates* $\operatorname{Hom}_{\underline{\mathrm{MF}}_T(h)}\left((\Phi,\Psi),(\Phi',\Psi')\right)$ *for all matrix factorizations* (Φ,Ψ)*,* (Φ',Ψ') *of* h*.*

Proof. Differentiate $\Phi\Psi = h\cdot 1_{\mathcal{F}}$ and $\Psi\Phi = h\cdot 1_{\mathcal{G}}$ by the Leibniz rule, and compose. □

Corollary 3.2. *Let* N *and* N' *be MCM* A*-modules. Then* (u,v) *annihilates* $\underline{\operatorname{Hom}}_A(N,N')$*.*

Proof. Since f involves neither u nor v, we have $\frac{\partial}{\partial u}(f+uv) = v$ and $\frac{\partial}{\partial v}(f+uv) = u$. □

Remark 3.3. Lemma 3.1 is not new. Its conclusion goes back at least to Buchweitz [Buc21, Corollary 7.8.7], who showed that for a complete intersection, the Jacobian ideal annihilates Tate cohomology, using the Kähler different. H.-J. Wang [Wan94, Theorem 5.4] and Iyengar-Takahashi [IT21, Theorem 4.2] extend it beyond complete intersections using the Noether different, and Herzog-Popescu [HP97, Proposition 2.5] state it for $K[\![\underline{x}]\!]/(f)$, K any field. The proof by differentiating appears in [KST07, Lemma 2.4], [Dyc11, §4.4 and Lemma 4.5] and [Mur13, Lemma 6.3]. I learned it from Özgür Esentepe, who used it to show [Ese20, Remark 3.1] that the Jacobian ideal is contained in the cohomological annihilator.

Proposition 3.4 ([PP96, Lemma 3.14], [Sol89, Prop. 3.1])**.** *The functor* $(-)^{\maltese}\colon \underline{\mathrm{MF}}_S(f) \longrightarrow \underline{\mathrm{MF}}_B(f+uv)$ *is fully faithful.*

Proof. This is [PP96, Lemma 3.14], in the setting of Remark 2.5. Earlier, Solberg proved full faithfulness of $F(\varphi,\psi) = \left(\left[\begin{smallmatrix} u & \varphi \\ \psi & -v \end{smallmatrix}\right], \left[\begin{smallmatrix} v & \varphi \\ \psi & -u \end{smallmatrix}\right]\right)$ for $S = k[\![z_1,\dots,z_r]\!]$; his proof is a computation with block matrices that uses only that $B = S[\![u,v]\!]$. The morphism $\left(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], 1\right) : F(\varphi,\psi) \longrightarrow (\psi,\varphi)^{\maltese} \cong \Sigma(\varphi,\psi)^{\maltese}$ is an isomorphism, natural in (φ,ψ). □

Let $\mathcal{I}$ be the essential image of $(-)^{\maltese}$ in $\mathrm{D_{sg}}(A)$.

Lemma 3.5. *The subcategory* $\mathcal{I}$ *is a thick subcategory of* $\mathrm{D_{sg}}(A)$*: it is closed under isomorphisms, finite direct sums,* $\Sigma^{\pm 1}$*, cones and direct summands.*

Proof. By Lemma 2.3(iii), $(-)^{\maltese}$ commutes with $\Sigma^{\pm 1}$ and with cones, and by Proposition 3.4 every morphism $X^{\maltese} \longrightarrow Y^{\maltese}$ is $\theta^{\maltese}$ for some $\theta\colon X \longrightarrow Y$; so $\mathcal{I}$ is closed under cones. Likewise every idempotent endomorphism of $X^{\maltese}$ is $e^{\maltese}$ for an idempotent e of X. Since idempotents split in $\mathrm{D_{sg}}(R)$, e is the projection onto a direct summand X' of X, and the corresponding summand of $X^{\maltese}$ is $(X')^{\maltese}$. □

Proposition 3.6. *Every object of* $\mathrm{D_{sg}}(A)$ *lies in* $\mathcal{I}$*. Equivalently, for every MCM* A*-module* N *there is a MCM* R*-module* M *with* $N \cong M^{\maltese}$ *in* $\underline{\mathrm{MCM}}(A)$*, and for* N *stable this is an isomorphism of modules.*

Proof. First, every MCM R-module M, viewed as an A-module via $A \twoheadrightarrow R$, lies in $\mathcal{I}$. To see this, set $N = M^{\maltese}$. By Lemma 2.3(i), $N^{\flat\flat} \cong M \oplus \mathrm{syz}_1^R M$. Since u,v is a regular sequence on N, the Koszul complex

$$0 \longrightarrow N \xrightarrow{\left[\begin{smallmatrix} -v \\ u \end{smallmatrix}\right]} N \oplus N \xrightarrow{[u\ v]} N \longrightarrow 0$$

is quasi-isomorphic to $M \oplus \mathrm{syz}_1^R M$. It is built from N by cones, so lies in $\mathcal{I}$, and hence so does its summand M.

Now let N be a stable MCM A-module, representing an arbitrary object of $\mathrm{D_{sg}}(A)$. By Corollary 3.2, u and v act as 0 on every $\underline{\mathrm{Hom}}_A(\Sigma^i N, \Sigma^j N)$. The Koszul complex on u,v over N is acyclic, so in $\mathrm{D_{sg}}(A)$

$$N^{\flat\flat} \cong \mathsf{cone}\left(\mathsf{cone}\left(N \xrightarrow{u} N\right) \xrightarrow{v} \mathsf{cone}\left(N \xrightarrow{u} N\right)\right).$$

Both maps are zero in $\mathrm{D_{sg}}(A)$, and the cone of a zero map $X \longrightarrow Y$ is $Y \oplus \Sigma X$. Hence

$$N^{\flat\flat} \cong N \oplus (\Sigma N)^2 \oplus \Sigma^2 N$$

in $\mathrm{D_{sg}}(A)$. So N is a direct summand of the MCM R-module $N^{\flat\flat}$, which lies in $\mathcal{I}$ by the first paragraph, and N lies in $\mathcal{I}$. □

Proof of Theorem 2.4. The triangulated functor $(-)^{\maltese}$ is fully faithful by Proposition 3.4 and dense by Proposition 3.6. □

Remark 3.7. Pfister and Popescu prove density differently [PP96, Cor. 3.7, Lemmas 3.8 and 3.13]. For a unit ϵ of S, A is finite and free over $S[\![v-\epsilon u]\!]$, and they show that the Noether different of this extension, which contains $v+\epsilon u$, annihilates $\mathrm{Ext}^1_A(N,-)$ for MCM N. Two units ϵ and $\epsilon+1$ then give u

and v; when k is the field with two elements there is no such pair, and they first pass to an extension of S with infinite residue field. They then split syzygies one variable at a time, over $A/uA = R[\![v]\!]$ and over A, and identify the result by row and column operations (compare Remark 4.3). Here u and v come directly from ∂_v and ∂_u, and one Koszul splitting in $\mathrm{D_{sg}}(A)$ handles both.

4. Explicit matrices

A MCM R-module M is naturally an A-module via $A \twoheadrightarrow A/(u,v) = R$. This section computes the MCM A-module $\operatorname{syz}_2^A M$, which represents M in $\mathrm{D_{sg}}(A)$, by an explicit resolution. The main result, Theorem 4.2, uses neither Proposition 3.4 nor Proposition 3.6.

We need one lemma, which will also be relevant in analyzing Example 5.3.

Lemma 4.1. *Let T be a regular ring, $h \in T$, and D a derivation of T. Then $T/(h)$ is regular at every prime ideal not containing Dh. In particular* $\operatorname{Sing} A \subseteq V(u,v)$*, and A is a normal domain.*

Proof. Let $\mathfrak{q}$ be a prime of T with $h \in \mathfrak{q}$ and $Dh \notin \mathfrak{q}$, and extend D to $T_\mathfrak{q}$. If $h \in \mathfrak{q}^2 T_\mathfrak{q}$, write $h = \sum_i a_i b_i$ with $a_i, b_i \in \mathfrak{q} T_\mathfrak{q}$; then $Dh = \sum_i ((Da_i)b_i + a_i(Db_i)) \in \mathfrak{q} T_\mathfrak{q}$, a contradiction. So $h \notin \mathfrak{q}^2 T_\mathfrak{q}$, and $T_\mathfrak{q}/(h)$ is regular.

For A, take $T = B$, $h = f + uv$ and $D = \frac{\partial}{\partial v}, \frac{\partial}{\partial u}$, with $Dh = u, v$. Since u, v is A-regular, (u,v) has height two, so A satisfies (R_1); it is Cohen-Macaulay, so satisfies (S_2). By Serre's criterion A is normal, and it is a domain since it is local. □

Theorem 4.2. *Let M be a stable MCM R-module, regarded as an A-module via $A \twoheadrightarrow R$. Then*

$$\operatorname{syz}_2^A M \cong \left(M \oplus \operatorname{syz}_1^R M\right)^{\divideontimes}, \tag{4.1}$$

and neither side has a non-zero free summand.

Proof. Let $(\varphi\colon G \longrightarrow F, \psi\colon F \longrightarrow G)$ be a reduced matrix factorization of f over S with $\operatorname{cok}(\varphi,\psi) = M$, and let n be the size of φ. As in [LW12, Notation 8.13], set $\widetilde{F} = A \otimes_S F$ and $\widetilde{G} = A \otimes_S G$, each isomorphic to A^n, and use φ and ψ also for the induced maps between them.

Since $M \cong F/\varphi(G)$ is killed by u and v, we have $M \cong \widetilde{F}/\left(\varphi(\widetilde{G}) + u\widetilde{F} + v\widetilde{F}\right)$, so there is an exact sequence

$$\widetilde{G} \oplus \widetilde{F} \oplus \widetilde{F} \xrightarrow{\alpha} \widetilde{F} \longrightarrow M \longrightarrow 0, \qquad \alpha = [\,\varphi \ u1_F \ v1_F\,],$$

and $\operatorname{syz}_2^A M = \ker \alpha$. Set

$$\beta = \begin{bmatrix} u & \psi & 0 & -v \\ -\varphi & v & v & 0 \\ 0 & 0 & -u & \varphi \end{bmatrix} : \widetilde{G} \oplus \widetilde{F} \oplus \widetilde{F} \oplus \widetilde{G} \longrightarrow \widetilde{G} \oplus \widetilde{F} \oplus \widetilde{F}.$$

Then $\alpha\beta = 0$. To show $\ker \alpha \subseteq \operatorname{im} \beta$, let $(W,X,Y) \in \ker \alpha$ and lift to $\hat{W}, \hat{X}, \hat{Y} \in B^n$, so that $\varphi(\hat{W}) + u\hat{X} + v\hat{Y} = gH$ for some $H \in B^n$, where $g = f + uv$. Since $gH = \varphi\psi(H) + uvH$,

$$\varphi(\hat{W} - \psi(H)) + u(\hat{X} - vH) + v\hat{Y} = 0$$

in B^n. The difference between the two lifts is in the image of the second column of β, so we may assume that

$$\varphi(W) + uX + vY = 0 \tag{4.2}$$

holds in B^n. Setting $u = v = 0$ gives $\varphi(W) = 0$ in S^n; since φ is injective over S, $W \in (u,v)B^n$. Write $W = uW_1 + vW_2$. Then $u(\varphi(W_1) + X) + v(\varphi(W_2) + Y) = 0$, and since u, v is a regular sequence on B^n there is $Z \in \widetilde{F}$ with $\varphi(W_1) + X = vZ$ and $\varphi(W_2) + Y = -uZ$. So

$$(W,X,Y) = (uW_1, -\varphi(W_1), 0) + (0, vZ, -uZ) + (vW_2, 0, -\varphi(W_2))$$

lies in the image of the first, third, and fourth columns of β.

Now set

$$\gamma = \begin{bmatrix} \psi & v & v & 0 \\ -u & \varphi & 0 & v \\ 0 & 0 & \varphi & -v \\ 0 & 0 & u & \psi \end{bmatrix} : \widetilde{F} \oplus \widetilde{G} \oplus \widetilde{G} \oplus \widetilde{F} \longrightarrow \widetilde{G} \oplus \widetilde{F} \oplus \widetilde{F} \oplus \widetilde{G}.$$

Then $\beta\gamma = 0$, so β factors through a surjection $\pi\colon \operatorname{cok}\gamma \longrightarrow \operatorname{im}\beta = \operatorname{syz}_2^A M$. The two diagonal $2n \times 2n$ blocks of γ are the matrices of Definition 2.2 in reverse order: with $(\Phi, \Psi) = (\varphi, \psi)^{\maltese}$,

$$\gamma = \begin{bmatrix} \Psi & v1 \\ 0 & \Phi \end{bmatrix}.$$

For $\partial = \frac{\partial}{\partial u}$ we have $\partial(f + uv) = v$, and the second equation of Lemma 3.1 reads $(\partial\Psi)\Phi + \Psi(\partial\Phi) = v1$. Hence

$$\begin{bmatrix} 1 & -\partial\Psi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Psi & v1 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} 1 & -\partial\Phi \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \Psi & 0 \\ 0 & \Phi \end{bmatrix}, \qquad \partial\Psi = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \partial\Phi = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

a change of basis defined and invertible over any ring. Therefore

$$\operatorname{cok}\gamma \cong \operatorname{cok}\Psi \oplus \operatorname{cok}\Phi = \operatorname{syz}_1^A\left(M^{\maltese}\right) \oplus M^{\maltese} \cong \left(\operatorname{syz}_1^R M\right)^{\maltese} \oplus M^{\maltese},$$

a MCM A-module without free summands. Since A is a normal domain (Lemma 4.1) and $\det\gamma = (f + uv)^{2n}$, [Eis80, Prop. 5.6] shows that $\operatorname{cok}\gamma$ has rank $2n$.

Finally, M has rank 0 over A, so the exact sequences $0 \longrightarrow \operatorname{syz}_1^A M \longrightarrow \widetilde{F} \longrightarrow M \longrightarrow 0$ and $0 \longrightarrow \operatorname{syz}_2^A M \longrightarrow \widetilde{G} \oplus \widetilde{F} \oplus \widetilde{F} \longrightarrow \operatorname{syz}_1^A M \longrightarrow 0$ give $\operatorname{rank}\operatorname{syz}_1^A M = n$ and $\operatorname{rank}\operatorname{syz}_2^A M = 2n$. Thus π is a surjection between torsion-free modules of the same rank, hence an isomorphism. □

Remark 4.3. Pfister and Popescu show by row and column operations [PP96, Lemma 3.13] that $\operatorname{syz}_1^A\left(\operatorname{syz}_1^{A/uA}(M)\right) \cong \left(M \oplus \operatorname{syz}_1^R M\right)^{\maltese}$ modulo free summands. Since A/uA has projective dimension 1 over A, it is zero in $\mathrm{D_{sg}}(A)$, so $M \cong \Sigma\operatorname{syz}_1^{A/uA} M$ there and the left side is $\operatorname{syz}_2^A M$ in $\underline{\mathrm{MCM}}\,A$. Thus their Lemma 3.13 gives Theorem 4.2 up to free direct summands. The proof above resolves M over A directly, and shows that neither side has a free summand.

The change of basis in the proof of Theorem 4.2, which makes the block-triangular matrix γ block-diagonal, uses the partial derivatives of (Φ, Ψ). These cannot be replaced by a polynomial expression in Φ and Ψ:

Proposition 4.4. *Let f, u, v, Φ, Ψ be indeterminates over $\mathbb{Z}$ and let*

$$\mathcal{U} = \mathbb{Z}[f,u,v]\langle\Phi,\Psi\rangle/(\Phi\Psi - (f+uv), \Psi\Phi - (f+uv))$$

be the generic algebra representing the data of a matrix factorization of $f + uv$. Then

$$v \notin \Psi\mathcal{U} + \mathcal{U}\Phi.$$

Consequently there are no matrices X and Y, given by universal polynomial formulas in f, u, v, Φ, and Ψ with integer coefficients, such that $\left[\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} \Psi & v1 \\ 0 & \Phi \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & Y \\ 0 & 1 \end{smallmatrix}\right]$ is block diagonal.

Proof. The algebra $\mathcal{U}$ is free over $\mathbb{Z}[f,u,v]$ on the basis $\{1, \Phi, \Psi, \Phi^2, \Psi^2, \ldots\}$, with multiplication determined by

$$\Phi\Phi^a = \Phi^{a+1}, \qquad \Psi\Psi^a = \Psi^{a+1}, \qquad \Phi\Psi^a = (f+uv)\Psi^{a-1}, \qquad \Psi\Phi^a = (f+uv)\Phi^{a-1}$$

for $a \geqslant 1$. So in every element of $\Psi\mathcal{U} + \mathcal{U}\Phi$ the coefficient of 1 lies in $(f + uv)\mathbb{Z}[f,u,v]$, and $v \notin (f+uv)$. □

The next corollary completes Lemma 2.3.

Corollary 4.5. *Let N be a stable MCM A-module. Then*

$$\left(N^{\flat\flat}\right)^{\divideontimes} \cong N \oplus \operatorname{syz}_1^A N;$$

in particular N is a direct summand of $\left(N^{\flat\flat}\right)^{\divideontimes}$.

Proof. By Proposition 3.6, $N \cong M^{\divideontimes}$ for a stable MCM R-module M, and by Lemma 2.3

$$\left(N^{\flat\flat}\right)^{\divideontimes} \cong \left(M \oplus \operatorname{syz}_1^R M\right)^{\divideontimes} \cong M^{\divideontimes} \oplus \operatorname{syz}_1^A\left(M^{\divideontimes}\right) = N \oplus \operatorname{syz}_1^A N. \qquad \square$$

5. The double branched cover

Consider the *double branched cover* and its iterate,

$$R^\sharp = S[\![z]\!]/(f+z^2), \qquad R^{\sharp\sharp} = S[\![z,w]\!]/(f+z^2+w^2).$$

When 2 is invertible and S contains $i=\sqrt{-1}$, the change of variables $u=z+iw$, $v=z-iw$ identifies $R^{\sharp\sharp}$ with $A=R^{\divideontimes}$. In equal characteristic 2 the two rings are not isomorphic in general. There are functors

$$(-)^\sharp\colon \operatorname{MCM}(R) \longrightarrow \operatorname{MCM}(R^\sharp), \quad M^\sharp = \operatorname{syz}_1^{R^\sharp} M, \qquad (-)^\flat\colon \operatorname{MCM}(R^\sharp) \longrightarrow \operatorname{MCM}(R), \quad N^\flat = N/zN.$$

Proposition 5.1 ([Knö87, Lemma 2.5])**.** *Let M be a stable MCM R-module and N a stable MCM $R^\sharp$-module. Then*

$$\left(M^\sharp\right)^\flat \cong M \oplus \operatorname{syz}_1^R M \tag{5.1}$$

and, if in addition $\operatorname{char} k \neq 2$*,*

$$\left(N^\flat\right)^\sharp \cong N \oplus \operatorname{syz}_1^{R^\sharp} N. \tag{5.2}$$

Knörrer proves (5.2) by diagonalizing the matrix $\left[\begin{smallmatrix} -\varphi & -z \\ z & \varphi \end{smallmatrix}\right]$ by a change of basis of determinant 2. Lemma 3.1 gives (5.2) in $\underline{\operatorname{MCM}} R^\sharp$ by the argument of Proposition 3.6, and shows how the hypothesis enters.

Proposition 5.2. *Assume 2 is a unit in S, and let N be a MCM $R^\sharp$-module. Then z annihilates $\underline{\operatorname{Hom}}_{R^\sharp}(N,N')$ for every MCM $R^\sharp$-module N', and $\left(N^\flat\right)^\sharp \cong N \oplus \operatorname{syz}_1^{R^\sharp} N$ in $\underline{\operatorname{MCM}} R^\sharp$.*

Proof. Since f does not involve z, $\frac{\partial}{\partial z}(f+z^2) = 2z$, so by Lemma 3.1 $2z$, and hence z, annihilates $\underline{\operatorname{Hom}}_{R^\sharp}(N,N')$. As $R^\sharp/zR^\sharp = R$, the element z is a parameter on $R^\sharp$, hence regular on N, and the exact sequence $0 \longrightarrow N \xrightarrow{z} N \longrightarrow N^\flat \longrightarrow 0$ gives a triangle in $\mathrm{D}_{\mathrm{sg}}(R^\sharp)$ whose first map is zero. So $N^\flat \cong N \oplus \Sigma N$ in $\mathrm{D}_{\mathrm{sg}}(R^\sharp)$, and

$$\left(N^\flat\right)^\sharp = \operatorname{syz}_1^{R^\sharp} N^\flat \cong \Sigma^{-1} N \oplus N \cong \operatorname{syz}_1^{R^\sharp} N \oplus N$$

in $\underline{\operatorname{MCM}} R^\sharp$. $\square$

The same argument, with Corollary 3.2 in place of $\frac{\partial}{\partial z}(f+z^2) = 2z$, shows that $\operatorname{syz}_1^A(N/uN) \cong N \oplus \operatorname{syz}_1^A N$ in $\underline{\operatorname{MCM}} A$ for every MCM A-module N, in every characteristic; Pfister and Popescu prove this as an isomorphism of modules [PP96, Cor. 3.7(ii)].

When $2=0$ in S, Lemma 3.1 gives nothing for $R^\sharp$, and the following example shows that (5.2) fails, while the conclusions of Sections 2 and 4 hold for A.

Example 5.3. Let k be a field of characteristic 2, $S=k[\![x]\!]$, and $f=x^2$, so that $R=k[\![x]\!]/(x^2)$. Then R is a zero-dimensional (A_1) singularity, with exactly two indecomposable MCM (equivalently, finitely generated) modules, namely

$$R = \operatorname{cok}(x^2,1) \qquad \text{and} \qquad k = \operatorname{cok}(x,x).$$

The hyperbolic extension $A = k[\![x,u,v]\!]/(x^2+uv)$ is another (A_1) singularity, and is an isolated singularity in characteristic 2 as well, since $\operatorname{jac}(x^2+uv) = (2x,v,u) = (u,v)$ is (u,v,x)-primary.

The double branched cover $R^\sharp = k[\![x,z]\!]/(x^2+z^2)$ is not an isolated singularity: $x^2+z^2 = (x+z)^2$, so $R^\sharp \cong k[\![s,t]\!]/(s^2)$ is an (A_∞) singularity. It therefore has infinite CM type by Auslander's theorem [LW12, Thm 7.12]; indeed, the MCM modules over the one-dimensional (A_∞) singularity are classified in [BGS87, §4]: there are the two obvious matrix factorizations of size 1, and

$$\left(\begin{bmatrix} s & t^j \\ 0 & s \end{bmatrix}, \begin{bmatrix} s & -t^j \\ 0 & s \end{bmatrix}\right)$$

for every $j \geqslant 1$. The classification holds in every characteristic.

Proposition 5.4. *With the notation of the Example:*

(i) *A has exactly two indecomposable MCM modules, namely A and $\mathfrak{p} = (u,x)A$. Both lie in the image of $(-)^{\divideontimes}$: explicitly $A = R^{\divideontimes}$ and $\mathfrak{p} = k^{\divideontimes}$.*

(ii) *$\left(\mathfrak{p}^{\flat\flat}\right)^{\divideontimes} \cong \mathfrak{p} \oplus \operatorname{syz}_1^A \mathfrak{p}$ and $\operatorname{syz}_2^A k = \mathfrak{p} \oplus \mathfrak{p} = \left(k \oplus \operatorname{syz}_1^R k\right)^{\divideontimes}$, in accordance with Corollary 4.5 and Theorem 4.2.*

(iii) *(5.2) does not hold. Set $N = R^\sharp/(x+z)R^\sharp$; then N is a stable indecomposable MCM $R^\sharp$-module with $\operatorname{syz}_1^{R^\sharp} N \cong N$, but $\left(N^\flat\right)^\sharp$ is indecomposable. Hence $\left(N^\flat\right)^\sharp \ncong N \oplus \operatorname{syz}_1^{R^\sharp} N$.*

(iv) *The iterated double branched cover $R^{\sharp\sharp} \cong k[\![s,z,w]\!]/(s^2)$ has infinite CM type, so that in particular $\underline{\operatorname{MCM}} R \neq \underline{\operatorname{MCM}} R^{\sharp\sharp}$.*

Proof. By Corollary 2.6 the indecomposable non-free MCM A-modules correspond bijectively via $(-)^{\divideontimes}$ to the indecomposable non-free R-modules, of which there is only one. So A has exactly two indecomposable MCM modules: $A = R^{\divideontimes}$ itself, and

$$k^{\divideontimes} = \operatorname{cok}(x,x)^{\divideontimes} = \operatorname{cok}\begin{bmatrix} x & -v \\ u & x \end{bmatrix}, \tag{5.3}$$

which is the height-one prime $\mathfrak{p} = (u,x)$.

Setting $u = v = 0$ in the presentation matrix of (5.3) gives $\operatorname{diag}(x,x)$, so that $\mathfrak{p}^{\flat\flat} \cong k \oplus k$ and $\left(\mathfrak{p}^{\flat\flat}\right)^{\divideontimes} = \mathfrak{p} \oplus \mathfrak{p}$. Since $\mathfrak{p}$ is two-generated of rank one, $\operatorname{syz}_1^A \mathfrak{p}$ also has rank one, and is not A, so $\operatorname{syz}_1^A \mathfrak{p} \cong \mathfrak{p}$. Over R, $\operatorname{syz}_1^R k \cong k$, giving $\operatorname{syz}_2^A k = \left(k \oplus \operatorname{syz}_1^R k\right)^{\divideontimes} = \mathfrak{p} \oplus \mathfrak{p}$ as in Theorem 4.2.

Consider $N = R^\sharp/(x+z)R^\sharp$ over $R^\sharp \cong k[\![x,z]\!]/((x+z)^2)$. As $(x+z,x+z)$ is a reduced matrix factorization of size 1, N is a stable indecomposable MCM $R^\sharp$-module, and $\operatorname{syz}_1^{R^\sharp} N \cong N$ since the factorization is symmetric. Setting $z = 0$ gives $N^\flat = \operatorname{cok}(x,x) \cong k$ over R. Then

$$\left(N^\flat\right)^\sharp = \operatorname{cok}\begin{bmatrix} x & -z \\ z & x \end{bmatrix} = \operatorname{cok}\begin{bmatrix} x & z \\ z & x \end{bmatrix}$$

as $-1 = 1$. In the variables $s = x+z$, $t = x$, this becomes

$$\operatorname{cok}\begin{bmatrix} t & s+t \\ s+t & t \end{bmatrix} \cong \operatorname{cok}\begin{bmatrix} t & s+t \\ s & t \end{bmatrix} \cong \operatorname{cok}\begin{bmatrix} t & s \\ s & 0 \end{bmatrix} \cong \operatorname{cok}\begin{bmatrix} s & t \\ 0 & s \end{bmatrix}$$

by elementary row and column operations, all invertible in characteristic 2. By the classification in [BGS87, §4], the last cokernel is indecomposable, while $N \oplus \operatorname{syz}_1^{R^\sharp} N \cong N \oplus N$ is not.

Finally, $f + z^2 + w^2 = (x+z+w)^2$, so $\operatorname{Sing} R^{\sharp\sharp}$ has dimension 2, and $R^{\sharp\sharp}$ has infinite type by Auslander's theorem. Meanwhile R has just one indecomposable non-free MCM module, so the stable categories cannot be equivalent. □

Remark 5.5. Suppose $2 = 0$ in S. Then the derivatives of $f + z^2 + w^2$ with respect to z and w are $2z = 0$ and $2w = 0$, so Lemma 4.1 says nothing about $R^{\sharp\sharp}$. In fact $f + z^2 + w^2 = f + (z+w)^2$, and the change of variables $s = z + w$ gives

$$R^{\sharp\sharp} \cong S[\![s,w]\!]/(f + s^2) = R^\sharp[\![w]\!].$$

The singular locus of $R^{\sharp\sharp}$ thus has dimension one greater than that of $R^{\sharp}$, and in particular $R^{\sharp\sharp}$ is not an isolated singularity. By Auslander's theorem $R^{\sharp\sharp}$ has infinite CM type for every f. So $\underline{\mathrm{MCM}}R \neq \underline{\mathrm{MCM}}R^{\sharp\sharp}$ whenever R has finite CM type, and Proposition 5.4(iv) is the case $f = x^2$.

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