

CHARACTERISTIC-FREE KNÖRRER PERIODICITY

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Dedicated to the memory of Ragnar-Olaf Buchweitz

ABSTRACT. We show that the Knörrer functor induces an equivalence $\underline{\text{MCM}}(R) \simeq \underline{\text{MCM}}(A)$ of stable categories of maximal Cohen-Macaulay modules, where $R = S/(f)$ is a complete hypersurface ring and $A = S[[u, v]]/(f + uv)$ is the hyperbolic extension of R . No hypothesis is placed on the residue field or on its characteristic, f need not define an isolated singularity, and S need not contain a field. This is in contrast to the iterated double branched cover $R^\sharp = S[[z, w]]/(f + z^2 + w^2)$, which is stably equivalent to R only in characteristic not equal to 2. We also give a direct construction of a free resolution identifying $\text{syz}_2^A(M)$ with the image of $M \oplus \text{syz}_1^R M$ under the functor, and an example in characteristic two in which the corresponding statement for the double branched cover $S[[z]]/(f + z^2)$ fails.

1. INTRODUCTION

Let $(S, \mathfrak{n}, k)$ be a complete regular local ring, let f be a non-zero element of $\mathfrak{n}^2$, and let $R = S/(f)$ be the associated hypersurface ring. The *double branched cover*

$$R^\sharp = S[[z]]/(f + z^2)$$

plays a central role in the classification of hypersurface rings of finite Cohen-Macaulay type, that is, those having only a finite number of isomorphism classes of indecomposable maximal Cohen-Macaulay modules.

Write $\text{MCM}(R)$ for the category of maximal Cohen-Macaulay (MCM) R -modules and $\underline{\text{MCM}}(R)$ for its stable category, obtained by killing the morphisms factoring through a free module. The surjection $R^\sharp \twoheadrightarrow R$ defined by killing the class of z induces functors

$$\begin{aligned} (-)^\sharp: \text{MCM}(R) &\longrightarrow \text{MCM}(R^\sharp), & M^\sharp &= \text{syzy}_1^{R^\sharp} M \\ (-)^\flat: \text{MCM}(R^\sharp) &\longrightarrow \text{MCM}(R), & N^\flat &= N/zN \end{aligned}$$

where syzy_1 denotes the syzygy operator. The relationships between these functors are pleasingly concrete and symmetric [LW12, Props. 8.15, 8.18]: as long as M and N are stable (having no non-zero free direct summands) we have

$$(M^\sharp)^\flat \cong M \oplus \text{syzy}_1^R M \tag{1.1}$$

and if in addition $\text{char } k \neq 2$,

$$(N^\flat)^\sharp \cong N \oplus \text{syzy}_1^{R^\sharp} N. \tag{1.2}$$

In characteristic 2, the second statement is known to fail, see Section 7.

The control that (1.1) and (1.2) give over MCM R - and $R^\sharp$ -modules allows one to prove that finite Cohen-Macaulay type descends from $R^\sharp$ to R in general, and ascends from R to $R^\sharp$ in characteristic not 2.

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A feature of the double branched cover construction is that doing it twice yields an even more agreeable situation. Set

$$R^{\#} = (R^{\sharp})^{\sharp} = S[[z, w]]/(f + z^2 + w^2).$$

Then Knörrer's periodicity theorem is as follows.

Theorem 1.1 ([Knö87, Theorem 3.1]). *Suppose $S = k[[x_0, \dots, x_d]]$ with k algebraically closed of characteristic different from two. Then there is an equivalence of stable categories*

$$\underline{\text{MCM}}(R) \simeq \underline{\text{MCM}}(R^{\#}),$$

induced by an explicit operation on matrix factorizations.

In Knörrer's proof the "explicit operation on matrix factorizations" is most easily expressed after a change of variables: set $u = z + iw$, $v = z - iw$ so that $z^2 + w^2 = uv$. The resulting ring is isomorphic to $R^{\#}$ if $i = \sqrt{-1}$ exists and 2 is a unit, but is genuinely different in characteristic 2, so we give it its own notation:

$$R^{\times} = S[[u, v]]/(f + uv)$$

and call it the *hyperbolic extension* of R .

It is important to note that Knörrer's result is about $f + z^2 + w^2$, not $f + uv$. Beyond the change of variables, Knörrer's proof also uses (1.2), which is false in characteristic 2. Even further, [Knö87, Lemma 1.3] chooses a convergent power series over S for $(1+x)^{-1/2}$, which is invalid in characteristic 2.

What is known without a hypothesis on the characteristic is the following. Solberg [Sol89, Theorem 4.5] proved, for k an arbitrary field, that $R = k[[z_1, \dots, z_r]]/(f)$ has finite CM type if and only if $R^{\times}$ does; his argument runs through Auslander-Reiten theory and therefore needs f to define an isolated singularity, and it needs finiteness of the AR quiver in an essential way. Yoshino [Yos98] observed that the Knörrer functor is given by the tensor product with the fixed matrix factorization (u, v) of uv , and proved, for $S = K[[x_0, \dots, x_d]]$ with K an arbitrary field, that it carries indecomposable stable MCM R -modules to indecomposable A -modules, and that $M^{\times} \cong (M')^{\times}$ forces $M' \cong M$ or $M' \cong \text{syz}_1^R M$ [Yos98, Lemma 2.12, Theorem 3.7]. Orlov proves an equivalence of singularity categories for f and $f + xy$ over regular schemes over a field [Orl04, Thm 2.1]; his paper does not address the characteristic, and we do not rely on it. Dyckerhoff [Dyc11, §5.3] proved a characteristic-free equivalence of the associated categories of matrix factorizations, using compact generation by the stabilized residue field; his hypothesis, condition (B) of [Dyc11, §3.2], also implies isolated singularity.

Here is the first main result of this paper.

Theorem A. *Let $(S, \mathfrak{n}, k)$ be a complete regular local ring, let $0 \neq f \in \mathfrak{n}^2$, and set $R = S/(f)$ and $A = R^{\times} = S[[u, v]]/(f + uv)$. Then the Knörrer functor of Def. 2.1*

$$(-)^{\times} : \underline{\text{MCM}}(R) \longrightarrow \underline{\text{MCM}}(A)$$

is an equivalence of triangulated categories.

The fact that $(-)^{\times}$ is fully faithful is already in [Sol89]. The hypotheses of the main theorem there include equicharacteristic, finite CM type, and an isolated singularity, but the proof of full faithfulness is a direct computation with block matrices which holds in full generality. This seems to be overlooked in the literature. We describe Solberg's proof more fully in Remark 2.10.

Density of $(-)^{\times}$ is proved in Section 4. It rests on the following elementary observation (Lemma 3.1), which again already appears in the literature: for a matrix factorization (Φ, Ψ) of an element h in a ring T , any derivation ∂ of T applied to the equation $\Phi\Psi = h \cdot 1$ yields a null-homotopy for multiplication by ∂h . Hence the Jacobian ideal of $f + uv$ annihilates the stable homomorphism groups $\underline{\text{Hom}}_A$.

It follows that the Koszul complex on u and v with coefficients in a MCM A -module N is built from maps that vanish in the stable category, so that N is a direct summand of N^{bb} . See Prop. 4.6.

Combining the lemma with a criterion of the author and Dugas [DL21, Lemma 2.2] gives a one-variable statement worth recording separately, and in the form closest to Theorem 1.1.

Corollary 1.2 (Corollary 3.5). *Let $A = S[[u, v]]/(f + uv)$ and let N be a MCM A -module. Then for z either of u or v ,*

$$\text{syz}_1^A(N/zN) \cong N \oplus \text{syz}_1^A(N),$$

with no hypothesis on $\text{char } k$.

Our second main result is a module-theoretic identity which does not pass through the stable category at all. It is proved in Section 5 by explicitly constructing a free resolution, and it is logically independent of Theorem A.

Theorem B. *Let M be a MCM R -module with no free direct summands, and regard M as a A -module through $A \twoheadrightarrow A/(u, v) = R$. Then*

$$\text{syz}_2^A(M) \cong M^{\times} \oplus (\text{syz}_1^R M)^{\times} = (M \oplus \text{syz}_1^R M)^{\times},$$

and neither side has a free direct summand.

Theorem B is an independent confirmation of the periodicity statement; the only scalars in its proof are ± 1 . It also reconciles the definition of the functor $(-)^{\times}$ with that of $(-)^{\sharp} = \text{syz}_1^{R^{\sharp}}$, contradicting the claim in [LW12, p. 160] that “... there seems to be no nice correspondence between MCM R -modules and MCM $R^{\sharp}$ -modules.”

Section 6 continues the explicit matrix computations, addressing the question of exhibiting Proposition 4.6 as a concrete matrix equation. We prove first that no universal formula exists: there is no polynomial in the input data that clears the necessary off-diagonal entries (Prop. 6.3). We then formulate the derivation/homotopy argument used in Theorem A in matrix language to give an independent proof of Theorem B.

Section 7 contains a minimal example of the failure of (1.2) and compares it with Theorems A and B.

Acknowledgements. This paper grew out of an extended exchange with the LLM Claude (Opus 5) about whether the characteristic $\neq 2$ hypothesis in Knörrer periodicity is essential. The model pointed out the key computations in Lemma 3.1 and Prop. 4.6, namely that partial derivatives form null-homotopies of u and v on the nose and therefore the Koszul complex splits in the singularity category. The model also compiled an initial draft. The author is responsible for all statements, proofs, and errors.

I was asked the initial question — Is there a direct characteristic-free proof of the equivalence $\text{MF}(f) \simeq \text{MF}(f + uv)$? — by Osamu Iyama on November 12, 2017, during a very pleasant and productive visit to Nagoya. My friend and collaborator Ragnar-Olaf Buchweitz had just passed away the day before. I have returned to the question many times over the years since, always with the feeling that Ragnar would have had deeper insight into the question than I was able to. I am pleased that the resulting proof uses so many of Ragnar’s favorite tools.

Notation. All rings are commutative and Noetherian and all modules are finitely generated unless stated otherwise. We write $\text{syz}_i^{\Gamma}(X)$ for an i th syzygy of X over Γ ; it is uniquely defined up to isomorphism if Γ is a local ring. A MCM module is *stable* if it has no non-zero free direct summand; a matrix factorization is *reduced* if all of its entries lie in the maximal ideal, and this happens if and only if its cokernel is stable [LW12, 8.35]. We use $\text{D}_{\text{sg}}(\Gamma) = \text{D}^b(\text{mod } \Gamma)/\text{perf}(\Gamma)$ for the singularity category.

2. MATRIX FACTORIZATIONS AND THE KNÖRRER FUNCTOR

Throughout the paper, $(S, \mathfrak{n}, k)$ is a complete regular local ring of dimension $d + 1$, f is a non-zero element of $\mathfrak{n}^2$, and we set

$$R = S/(f), \quad B = S[[u, v]], \quad A = R^{\mathbf{x}} = B/(f + uv).$$

We sometimes write $g = f + uv$ for brevity, so that $A = B/(g)$. No hypothesis is placed on k anywhere below; in particular we do not assume that S contains a coefficient field.

Recall [LW12, 8.2] that a *matrix factorization* of f over S is a pair $(\varphi: G \rightarrow F, \psi: F \rightarrow G)$ of homomorphisms of free S -modules of the same finite rank with

$$\psi\varphi = f \cdot 1_G \quad \text{and} \quad \varphi\psi = f \cdot 1_F,$$

and that $\text{cok}(\varphi, \psi) := \text{cok } \varphi$ is then a MCM R -module. Eisenbud's theorem [Eis80, §6] says that cok induces a bijection between the reduced matrix factorizations of f up to equivalence and the stable MCM R -modules up to isomorphism; see Lemma 2.6 for a precise statement.

It is well-known that reversing the pair (φ, ψ) gives the syzygy over R :

$$\text{cok}(\psi, \varphi) \cong \text{syz}_1^R \text{cok}(\varphi, \psi). \tag{2.1}$$

In particular $\text{syzy}_2^R \cong \text{id}$ on stable MCM R -modules.

Definition 2.1 ([LW12, 8.31]).

- (1) For $M = \text{cok}(\varphi: G \rightarrow F, \psi: F \rightarrow G)$ a MCM R -module, set

$$M^{\mathbf{x}} = \text{cok} \left(\begin{bmatrix} \varphi & -v \cdot 1_F \\ u \cdot 1_G & \psi \end{bmatrix}, \begin{bmatrix} \psi & v \cdot 1_G \\ -u \cdot 1_F & \varphi \end{bmatrix} \right),$$

a MCM A -module.

- (2) For N a MCM A -module, define

$$N^{bb} = N/(u, v)N,$$

a MCM R -module.

The assertion that $M^{\mathbf{x}}$ is a MCM A -module is equivalent to the easily-verified fact that the displayed matrices in the definition do indeed comprise a matrix factorization of $f + uv$. The assertion that N^{bb} is MCM over R follows from the fact that u, v form a regular sequence on A and thus on every MCM A -module.

Remark 2.2. The functor $(-)^{\mathbf{x}}$ appears previously in both [Knö87, §3] and [Sol89, §2]. Knörrer's version uses the matrix $\begin{bmatrix} u & \psi \\ \varphi & -v \end{bmatrix}$, which is an elementary block-row operation away from our definition, so is isomorphic to ours. Solberg's functor differs by a syzygy; we will compare the two more precisely in Lemma 2.8 below, since we require Solberg's results for the proof of Theorem A.

For later use we record here the rank of $M^{\mathbf{x}}$.

Lemma 2.3. *Let n be the size of the reduced matrix factorization (φ, ψ) . Then $(\text{cok } \varphi)^{\mathbf{x}}$ has rank n as an A -module.*

Proof. By [Eis80, Prop. 5.6], the rank of $\text{cok } \varphi$ as an R -module is equal to the largest p with $\det \varphi = rf^p$ for some $r \in R$. Applying this to $(\text{cok } \varphi)^{\mathbf{x}}$ over A , we must compute the order of $g = f + uv$ in the determinant of the block matrix $\begin{bmatrix} \varphi & -v \\ u & \psi \end{bmatrix}$. Since the blocks all commute, this is $\det(\varphi\psi + uv \cdot 1_n) = \det(g \cdot 1_n) = g^n$. $\square$

The next lemma is two-thirds of [LW12, Lemma 8.32]. Both parts are elementary but we include the proofs to verify that they are characteristic-free.

Lemma 2.4. *Let M be a MCM R -module. Then*

- (i) $(M^\times)^{bb} \cong M \oplus \text{syz}_1^R M$; and
- (ii) $(\text{syz}_1^R M)^\times \cong \text{syz}_1^A(M^\times)$.

Proof. For (i), write $M^\times = \text{cok}_B \Phi$ with $\Phi = \begin{bmatrix} \varphi & -v \\ u & \psi \end{bmatrix}$. Tensoring the presentation $B^{2n} \xrightarrow{\Phi} B^{2n} \longrightarrow M^\times \longrightarrow 0$ with $S = B/(u, v)$ is right exact, so

$$M^\times / (u, v) M^\times = \text{cok} \begin{bmatrix} \varphi & 0 \\ 0 & \psi \end{bmatrix} = \text{cok}(\varphi, \psi) \oplus \text{cok}(\psi, \varphi) \cong M \oplus \text{syz}_1^R M.$$

For (ii), we have $\text{syz}_1^A(M^\times) = \text{cok} \begin{bmatrix} \psi & v \\ -u & \varphi \end{bmatrix}$, while $(\text{syz}_1^R M)^\times = \text{cok} \begin{bmatrix} \psi & -v \\ u & \varphi \end{bmatrix}$. Conjugating by the invertible (over any ring) matrix $\text{diag}(1, -1)$ transforms the second into the first, so the cokernels are isomorphic. $\square$

Part (ii) of the Lemma above says that $(-)^\times$ commutes with the syzygy functors over R and A . In order to show that $(-)^\times$ is actually an exact functor on the triangulated category $\underline{\text{MCM}}(R)$, we must recall the basics of the triangulated structure via that on the homotopy category of matrix factorizations $\underline{\text{MF}}_S(f)$.

Definition 2.5. Let $X = (\varphi: G \longrightarrow F, \psi: F \longrightarrow G)$, $X' = (\varphi': G' \longrightarrow F', \psi': F' \longrightarrow G')$ be matrix factorizations of an element f of a ring S .

- (1) A *morphism* $(\alpha, \beta): X \longrightarrow X'$ is a pair of homomorphisms of free modules $\alpha: F \longrightarrow F'$ and $\beta: G \longrightarrow G'$ satisfying $\alpha\varphi = \varphi'\beta$ and $\beta\psi = \psi'\alpha$.
- (2) The morphism (α, β) is *null-homotopic* if there exist $s: F \longrightarrow G'$ and $t: G \longrightarrow F'$ with

$$\alpha = \varphi' s + t \psi \quad \text{and} \quad \beta = \psi' t + s \varphi.$$

- (3) The *suspension* of X is $\Sigma X = (-\psi, -\varphi)$ and that of (α, β) is (β, α) .
- (4) The *cone* of (α, β) is

$$\text{cone}(\alpha, \beta) = \left(\begin{bmatrix} \varphi' & \alpha \\ 0 & -\psi \end{bmatrix}, \begin{bmatrix} \psi' & \beta \\ 0 & -\varphi \end{bmatrix} \right).$$

It is again a matrix factorization of f .

- (5) The morphisms

$$\iota = (\iota_1, \iota_2) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right): X' \longrightarrow \text{cone}(\alpha, \beta)$$

$$\pi = (\pi_1, \pi_2) = \left(\begin{bmatrix} 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \end{bmatrix} \right): \text{cone}(\alpha, \beta) \longrightarrow \Sigma X$$

give the *standard triangle* $X \xrightarrow{(\alpha, \beta)} X' \xrightarrow{\iota} \text{cone}(\alpha, \beta) \xrightarrow{\pi} \Sigma X$.

- (6) The distinguished triangles in $\underline{\text{MF}}_S(f)$ are those isomorphic to a standard triangle.

Lemma 2.6 ([Buc21, Thm. 4.4.1], [Eis80, §6]). *With the definitions above, $\underline{\text{MF}}_S(f)$ is a triangulated category, and cok induces an equivalence of triangulated categories $\underline{\text{MF}}_S(f) \simeq \underline{\text{MCM}}(R)$.* $\square$

Proposition 2.7. *For a morphism $(\alpha, \beta): X \longrightarrow X'$ of matrix factorizations of f over S , set*

$$(\alpha, \beta)^\times = \left(\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix} \right).$$

- (i) *This makes $(-)^\times$ into a functor from matrix factorizations of f over S to those of $f + uv$ over B , and the functor carries null-homotopic morphisms to null-homotopic morphisms.*
- (ii) *$(-)^\times$ commutes with suspension on both objects and morphisms: $(\Sigma X)^\times = \Sigma(X^\times)$ and $(\Sigma(\alpha, \beta))^\times = \Sigma((\alpha, \beta)^\times)$.*

- (iii) $(-)^{\mathbf{x}}$ commutes with cones; specifically, there is an isomorphism $\sigma: \text{cone}(\alpha, \beta)^{\mathbf{x}} \rightarrow \text{cone}((\alpha, \beta)^{\mathbf{x}})$ in $\mathbf{MF}_B(f + uv)$, both components of which are permutation matrices, such that $\sigma \iota^{\mathbf{x}} = \iota$ and $\pi \sigma = \pi^{\mathbf{x}}$.

In particular $(-)^{\mathbf{x}}$ gives a functor of triangulated categories $\mathbf{MF}_S(f) \rightarrow \mathbf{MF}_B(f + uv)$, equivalently $\mathbf{MCM}(R) \rightarrow \mathbf{MCM}(A)$.

The proof of the Proposition is elementary, but we include it to verify that it is valid over any S .

Proof. Write $X^{\mathbf{x}} = (\Phi: G \oplus F \rightarrow F \oplus G, \Psi: F \oplus G \rightarrow G \oplus F)$ and $X'^{\mathbf{x}} = (\Phi': G' \oplus F' \rightarrow F' \oplus G', \Psi': F' \oplus G' \rightarrow G' \oplus F')$ as in Definition 2.1. Also write $(A, B) = (\alpha, \beta)^{\mathbf{x}}$.

(i) We have $A\Phi = \Phi'B$ and $B\Psi = \Psi'A$ by a direct computation, so (A, B) is a morphism. Given a null-homotopy (s, t) for (α, β) , put $S = \text{diag}(s, t)$ and $T = \text{diag}(t, s)$; another direct computation shows that $\Phi'S + T\Psi = A$ and $\Psi'T + S\Phi = B$.

(ii) We have $\Sigma X = (-\psi: F \rightarrow G, -\varphi: G \rightarrow F)$, so by Definition 2.1

$$(\Sigma X)^{\mathbf{x}} = \left(\begin{bmatrix} -\psi & -v \\ u & -\varphi \end{bmatrix}, \begin{bmatrix} -\varphi & v \\ -u & -\psi \end{bmatrix} \right) = (-\Psi, -\Phi) = \Sigma(\Phi, \Psi).$$

On morphisms both sides send (α, β) to $(\text{diag}(\beta, \alpha), \text{diag}(\alpha, \beta))$.

(iii) Set

$$C = \text{cone}(\alpha, \beta) = \left(\begin{bmatrix} \varphi' & \alpha \\ 0 & -\psi \end{bmatrix}: G' \oplus F \rightarrow F' \oplus G, \begin{bmatrix} \psi' & \beta \\ 0 & -\varphi \end{bmatrix}: F' \oplus G \rightarrow G' \oplus F \right).$$

Then $C^{\mathbf{x}}$ consists of the pair of matrices

$$\begin{bmatrix} \varphi' & \alpha & -v \\ 0 & -\psi & -v \\ u & & \psi' & \beta \\ & u & 0 & -\varphi \end{bmatrix}: G' \oplus F \oplus F' \oplus G \rightarrow F' \oplus G \oplus G' \oplus F,$$

$$\begin{bmatrix} \psi' & \beta & v \\ 0 & -\varphi & v \\ -u & & \varphi' & \alpha \\ & -u & 0 & -\psi \end{bmatrix}: F' \oplus G \oplus G' \oplus F \rightarrow G' \oplus F \oplus F' \oplus G$$

Meanwhile, $(\alpha, \beta)^{\mathbf{x}} = (A, B) = (\text{diag}(\alpha, \beta): F \oplus G \rightarrow F' \oplus G', \text{diag}(\beta, \alpha): G \oplus F \rightarrow G' \oplus F')$ and so

$$\text{cone}((\alpha, \beta)^{\mathbf{x}}) = \left(\begin{bmatrix} \Phi' & A \\ 0 & -\Psi \end{bmatrix}: G' \oplus F' \oplus F \oplus G \rightarrow F' \oplus G' \oplus G \oplus F, \right.$$

$$\left. \begin{bmatrix} \Psi' & B \\ 0 & -\Phi \end{bmatrix}: F' \oplus G' \oplus G \oplus F \rightarrow G' \oplus F' \oplus F \oplus G \right).$$

Swapping the second and third rows, and second and third columns, of each matrix in $C^{\mathbf{x}}$ yields the matrices in $\text{cone}(A, B)$. The permutation matrix σ accomplishing these swaps satisfies the listed equations.

For the last assertion, we compile the previous parts: By (i) the functor $(-)^{\mathbf{x}}$ induces an additive functor on homotopy categories, which by (ii) commutes with the suspension functors. By (iii), cones are carried to cones, whence standard triangles are carried to standard triangles. The same is true of distinguished triangles by definition. The statement for $\mathbf{MCM}$ follows from Lemma 2.6. $\square$

Solberg proves [Sol89, Prop. 3.1] that his version of $(-)^{\mathbf{x}}$ is fully faithful. To translate this to our context, we must make precise the relationship between the two versions.

Lemma 2.8. *Let (φ, ψ) be a matrix factorization of f of size n , let $M = \text{cok}(\varphi, \psi) \in \text{MCM}(R)$, and let $F(-)$ be the functor of [Sol89, §2], defined on objects by*

$$F(\varphi, \psi) = \left(\begin{bmatrix} u & \varphi \\ \psi & -v \end{bmatrix}, \begin{bmatrix} v & \varphi \\ \psi & -u \end{bmatrix} \right).$$

Put $\epsilon = \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}$. Then

- (i) $(\epsilon, 1)$ is an isomorphism $F(\varphi, \psi) \xrightarrow{\sim} (\psi, \varphi)^{\mathbf{x}}$ in $\text{MF}_B(f + uv)$, natural in (φ, ψ) ; and
- (ii) there is therefore a natural isomorphism of functors

$$\text{cok} \circ F \cong \text{syzy}_1^A \circ (-)^{\mathbf{x}} : \underline{\text{MCM}}(R) \longrightarrow \underline{\text{MCM}}(A),$$

where we define F on stable MCM R -modules in the natural way. In particular $(-)^{\mathbf{x}}$ is fully faithful if and only if F is.

Proof. It is straightforward to check that $F(\varphi, \psi)$ is indeed a matrix factorization of $f + uv$. A morphism $(\alpha, \beta) : (\Phi_1, \Psi_1) \longrightarrow (\Phi_2, \Psi_2)$ in $\text{MF}_B(f + uv)$ is a pair of matrices with $\alpha\Phi_1 = \Phi_2\beta$ and $\beta\Psi_1 = \Psi_2\alpha$; taking $(\alpha, \beta) = (\epsilon, 1)$ gives

$$\begin{aligned} \left(\epsilon \begin{bmatrix} u & \varphi \\ \psi & -v \end{bmatrix}, \begin{bmatrix} v & \varphi \\ \psi & -u \end{bmatrix} \epsilon \right) &= \left(\begin{bmatrix} \psi & -v \\ u & \varphi \end{bmatrix}, \begin{bmatrix} \varphi & v \\ -u & \psi \end{bmatrix} \right) \\ &= (\psi, \varphi)^{\mathbf{x}}. \end{aligned}$$

Since $\epsilon^2 = 1$, the morphism $(\epsilon, 1)$ is its own inverse. Naturality is checked by a similar calculation.

For (ii), we have

$$\begin{aligned} \text{cok} F(\varphi, \psi) &= \text{cok}(\psi, \varphi)^{\mathbf{x}} \\ &= \text{syzy}_1^R(M)^{\mathbf{x}} \\ &\cong \text{syzy}_1^A(M^{\mathbf{x}}), \end{aligned}$$

the isomorphism being Lemma 2.4(ii).

The last sentence is a formal consequence of the fact that cok is an equivalence and syzy_1^A an auto-equivalence. $\square$

Theorem 2.9. *The functor $(-)^{\mathbf{x}} : \underline{\text{MCM}}(R) \longrightarrow \underline{\text{MCM}}(A)$ is a fully faithful functor of triangulated categories.*

Proof. Exactness is Proposition 2.7. Proposition 3.1 of [Sol89] proves full faithfulness for F , and Lemma 2.8 transfers this property to $(-)^{\mathbf{x}}$. $\square$

Remark 2.10. The main results of [Sol89] are often cited as applying to equicharacteristic hypersurface rings of finite Cohen-Macaulay type. We describe his proof of full faithfulness briefly, to confirm that it does not depend on the characteristic nor even on the ring containing a field.

To show

$$\underline{\text{Hom}}_{\text{MF}_S(f)}((\varphi, \psi), (\varphi', \psi')) \cong \underline{\text{Hom}}_{\text{MF}_B(f + uv)}(F(\varphi, \psi), F(\varphi', \psi')),$$

injectivity is immediate from Lemma 2.8(i). For surjectivity, Solberg first identifies two families of morphisms between $F(\varphi, \psi)$ and $F(\varphi', \psi')$ that factor through the projective objects $(f + uv, 1)$ and $(1, f + uv)$. He then writes out the eight matrix equations that the blocks of an arbitrary morphism must satisfy, writes each block as a power series in u over $S[[v]]$, and subtracts morphisms from the two families successively. The result turns out to be of the form

$$\begin{pmatrix} \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}, \begin{bmatrix} 0 & \delta \\ \gamma & 0 \end{bmatrix} \end{pmatrix}$$

and satisfies $u\beta = -v\gamma$ and $-v\alpha = u\delta$. It follows that $\alpha = uP$ is a multiple of u and $\beta = vQ$ is a multiple of v . Since the resulting expressions for uP and $-uQ$ are then shown not to depend on u , he concludes $P = Q = 0$.

The operations in this argument are block-multiplication and writing block matrices over $S[[v]]$, and the conclusion follows as u, v is a regular sequence. No coefficients other than ± 1 arise.

3. THE JACOBIAN HOMOTOPIES

This section contains the elementary observation on derivations that we will use to establish density of the Knörrer functor $(-)^{\mathfrak{K}}$. The statement is not new; see Remark 3.3 for some of the history.

Lemma 3.1. *Let T be a commutative ring, $h \in T$, and let $(\Phi: \mathcal{G} \rightarrow \mathcal{F}, \Psi: \mathcal{F} \rightarrow \mathcal{G})$ be a matrix factorization of h over T . Let ∂ be any derivation of T , extended entrywise to matrices. Then*

$$(\partial\Phi)\Psi + \Phi(\partial\Psi) = (\partial h) \cdot 1_{\mathcal{F}} \quad \text{and} \quad (\partial\Psi)\Phi + \Psi(\partial\Phi) = (\partial h) \cdot 1_{\mathcal{G}}. \quad (3.1)$$

In particular the pair $(\partial\Psi, \partial\Phi)$ is a null-homotopy for multiplication by ∂h . It follows that ∂h annihilates the stable endomorphism ring of (Φ, Ψ) , and hence that ∂h annihilates $\underline{\text{Hom}}((\Phi, \Psi), (\Phi', \Psi'))$ for every matrix factorization (Φ', Ψ') of h . In particular the Jacobian ideal $\text{jac}(h)$ annihilates $\underline{\text{Hom}}_T$.

Proof. The equations (3.1) follow from differentiating $\Phi\Psi = h \cdot 1_{\mathcal{F}}$ and $\Psi\Phi = h \cdot 1_{\mathcal{G}}$ and obeying the Leibniz rule. Either equation implies that multiplication by ∂h factors through a free module. The last two assertions follow by composing. $\square$

Corollary 3.2. *Let N and N' be MCM A -modules. Then (u, v) annihilates $\underline{\text{Hom}}_A(N, N')$. Moreover (u, v) annihilates $\text{Ext}_A^1(N, Y)$ for every A -module Y .*

Proof. Apply Lemma 3.1 with $\partial = \frac{\partial}{\partial v}$ and $\partial = \frac{\partial}{\partial u}$, and observe that

$$\frac{\partial}{\partial u}(f + uv) = v \quad \text{and} \quad \frac{\partial}{\partial v}(f + uv) = u,$$

since f involves neither u nor v .

The last statement follows from the fact that multiplication by u , resp. v , factors through a free A -module, together with the functoriality of Ext . $\square$

Remark 3.3. Lemma 3.1 is not new, and indeed has appeared multiple times in the literature in a few different costumes. The earliest avatar I can find is [Buc21, Corollary 7.8.7], which states that if $f = (f_1, \dots, f_c)$ is a regular sequence in the maximal ideal of a regular local ring P , then the Jacobian ideal $\text{jac}(f)$ annihilates all Tate cohomology groups over the complete intersection ring $P/(f)$. This is more general than Lemma 3.1, both in allowing complete intersections of codimension greater than one and in covering all of Tate cohomology. The proof does not use derivations directly, instead using the Kähler (or Dedekind) different, which in this setting agrees with the Noether different, and citing [Kun86, Sätze 1 & 2].

Buchweitz's result, in the form “ $\text{jac}(T)$ annihilates $\text{Ext}_T^1(N, Y)$ when N is maximal Cohen-Macaulay” was extended beyond complete intersections by H.-J. Wang [Wan94, Theorem 5.4] and Iyengar-Takahashi [IT21, Theorem 4.2], again using (generalizations of) the Noether different.

The proof above, differentiating $\Phi\Psi = h \cdot 1$, has apparently been rediscovered several times. It appears in [KST07, Lemma 2.4], showing that $\underline{\text{Hom}}$ is a finitely-generated module over the Jacobian ring, hence is a finite-dimensional vector space when f defines an isolated singularity. It reappears in [Dyc11, §4.4 and Lemma 4.5] in the context of annihilating Tor , and in [Mur13, Lemma 6.3] in writing down a residue formula for a nondegenerate pairing on homotopy classes of matrix factorizations. I first learned of this proof from Özgür Esentepe, who used it to conclude [Ese20, Remark 3.1] that the Jacobian ideal is contained in the *cohomological annihilator*

$$\text{ca}(T) = \bigcup_n \bigcap_{N, N' \text{ MCM}} \text{Ann}_T \text{Ext}_T^n(N, N').$$

The significance of Lemma 3.1 for our purposes is twofold. Unlike some other similar statements, which deliver a *power* of an element that annihilates $\underline{\text{Hom}}$ or Ext^1 , Lemma 3.1 gives exponent 1. Secondly, for the hyperbolic extension $R^\times$, the two adjoined variables are themselves partial derivatives of $f + uv$ on the nose, whereas for the double branched cover the corresponding computation is $\frac{\partial}{\partial z}(f + z^2) = 2z$, which is a significant difference if 2 is not a unit in R or is even equal to 0. This asymmetry is what permits Lemma 4.2 below and makes [LW12, Prop. 8.18] fail in characteristic 2, as we show in Section 7.

We can now record the one-variable statement promised in the introduction. It is the analogue for the hyperbolic extension of Knörrer's Theorem 1.1 in the form given in [DL21], and unlike that theorem it needs no hypothesis on the characteristic.

Lemma 3.4 ([DL21, Lemma 2.2]). *Let Q be a commutative Noetherian ring, X a finitely generated Q -module, and z a non-zerodivisor on X . The following are equivalent.*

- (i) $\text{syz}_1^Q(X/zX) \cong X \oplus \text{syz}_1^Q(X)$;
- (ii) z annihilates $\text{Ext}_Q^1(X, \text{syz}_1^Q(X))$;
- (iii) z annihilates $\text{Ext}_Q^1(X, Y)$ for every finitely generated Q -module Y .

For the reader's convenience we recall the argument. There is a short exact sequence $0 \rightarrow \text{syz}_1^Q(X) \rightarrow \text{syz}_1^Q(X/zX) \rightarrow X \rightarrow 0$, obtained by pulling back a projective presentation of X along multiplication by z , and this sequence is the image of the element $z \cdot \text{id}_X$ under $\text{End}_Q(X) \rightarrow \text{Ext}_Q^1(X, \text{syz}_1^Q(X))$. So z kills that Ext group if and only if the sequence splits, which by Miyata's theorem [Miy67] happens if and only if the isomorphism in (i) holds.

Corollary 3.5. *Let N be a MCM A -module and let z be either of u or v . Then*

$$\text{syz}_1^A(N/zN) \cong N \oplus \text{syz}_1^A(N).$$

Proof. Apply Lemma 3.4, noting that z is a non-zerodivisor on N and annihilates $\text{Ext}_A^1(N, Y)$ for all Y by Corollary 3.2. $\square$

4. DENSITY

In this section we prove density of the Knörrer functor $(-)^*$. Keep the notation established thus far.

At this point it becomes convenient to pass from the stable module category $\underline{\text{MCM}}$ to the singularity category D_{sg} . Recall that this is the Verdier quotient $\text{D}_{\text{sg}}(\Gamma) = \text{D}^b(\text{mod } \Gamma) / \text{perf}(\Gamma)$, where $\text{perf}(\Gamma)$ consists of the perfect complexes over Γ , i.e. those which are isomorphic in $\text{D}(\text{mod } \Gamma)$ to a bounded complex of finitely generated projective modules. Buchweitz proves [Buc21, Thm 4.4.1]

$$\underline{\text{MCM}}(\Gamma) \simeq \text{D}_{\text{sg}}(\Gamma)$$

for Γ Gorenstein; in particular, this equivalence holds for both R and A . We denote by Σ the shift on D_{sg} , which is syz_1^{-1} on MCM modules.

The advantage of working in the singularity category is that the surjection $A \rightarrow R = A/(u, v)$ allows us to consider (MCM) R -modules as objects in $\text{D}_{\text{sg}}(A)$ directly.

We collect a few lemmas.

Lemma 4.1. *Viewed as a complex over A via $A \rightarrow R$, the ring R is a perfect complex, hence $R \cong 0$ in $\text{D}_{\text{sg}}(A)$. Furthermore, for every finitely generated R -module X , there is an isomorphism $X \cong \Sigma(\text{syz}_1^R X)$ in $\text{D}_{\text{sg}}(A)$. In particular if X is a stable MCM R -module then $\text{syz}_2^A X \cong X$ in $\text{D}_{\text{sg}}(A)$.*

Proof. Since u, v is a regular sequence in A and $A/(u, v) = R$, R is quasi-isomorphic to the Koszul complex $0 \rightarrow A \xrightarrow{\begin{bmatrix} -v \\ u \end{bmatrix}} A^2 \xrightarrow{[uv]} A \rightarrow 0$.

For the statements about a finitely generated R -module X , let $0 \rightarrow \text{syz}_1^R X \rightarrow F \rightarrow X \rightarrow 0$ be an exact sequence of R -modules with F finitely generated free. Considered in $D_{\text{sg}}(A)$ we have $F \cong 0$ by above, so the associated triangle gives $X \cong \Sigma(\text{syz}_1^R X)$. Applying this to $\text{syz}_1^R X$ in place of X gives $X \cong \Sigma^2(\text{syz}_2^R X)$, and since $\Sigma^{-2} = \text{syz}_2^A$ on $D_{\text{sg}}(A)$, we may conclude $\text{syz}_2^A X \cong \text{syz}_2^R X$. If X is stable MCM over R , then $\text{syz}_2^R X \cong X$. $\square$

Lemma 4.2. *The singular locus of A is contained in $V(u, v)$. Equivalently, $A_{\mathfrak{p}}$ is a regular local ring for $\mathfrak{p} \in \text{Spec } A$ if $\mathfrak{p} \not\supseteq (u, v)$.*

Proof. Let $\mathfrak{p} \in \text{Spec } A$ with $(u, v) \not\subseteq \mathfrak{p}$. At least one of u, v must lie outside $\mathfrak{p}$; by symmetry we may assume $u \notin \mathfrak{p}$.

Let $\mathfrak{q} \subseteq B$ be the preimage of $\mathfrak{p}$, so that $A_{\mathfrak{p}} = B_{\mathfrak{q}}/(g)B_{\mathfrak{q}}$ and the image of u is a unit in $B_{\mathfrak{q}}$. If $g \in \mathfrak{q}^2 B_{\mathfrak{q}}$, then we can write $g = \sum_i a_i b_i$ with all $a_i, b_i \in \mathfrak{q} B_{\mathfrak{q}}$; then

$$\frac{\partial}{\partial v} g = \sum_i \left(\left(\frac{\partial}{\partial v} a_i \right) b_i + a_i \left(\frac{\partial}{\partial v} b_i \right) \right) \in \mathfrak{q} B_{\mathfrak{q}}$$

since every term contains a factor from $\mathfrak{q} B_{\mathfrak{q}}$. But $\partial g / \partial v = u \notin \mathfrak{q} B_{\mathfrak{q}}$, a contradiction. This implies $g \notin \mathfrak{q}^2 B_{\mathfrak{q}}$ and $A_{\mathfrak{p}}$ regular. $\square$

Lemma 4.3. *The hyperbolic extension $A = B/(f + uv)$ is a normal domain.*

Proof. Since u, v form a regular sequence in A , the ideal (u, v) has height two. Then Lemma 4.2 shows that $A_{\mathfrak{p}}$ is regular for every prime $\mathfrak{p}$ of height at most one, so A satisfies (R_1) . Being a hypersurface, A is Cohen-Macaulay and so satisfies (S_2) . By Serre's criterion A is a normal ring, and a domain since it is local. $\square$

Lemma 4.4. *Let M be a MCM R -module without free direct summands. Then M is indecomposable in $\text{MCM}(R)$ if and only if it is indecomposable in $\underline{\text{MCM}}(R)$.*

Proof. A nontrivial decomposition $M = M_1 \oplus M_2$ in $\text{MCM}(R)$ passes to one in $\underline{\text{MCM}}(R)$, still nontrivial since M and all summands of M are stable. The content is the other direction.

Let $\mathcal{P}_M \subseteq \text{End}_R(M)$ be the ideal of endomorphisms factoring through a free R -module; we will show that $\mathcal{P}_M$ is contained in the Jacobson radical of $\text{End}_R(M)$. Indeed, if M is indecomposable then $\text{End}_R(M)$ is a nc-local ring (in the terminology of [LW12, §1.1], meaning that the quotient $\text{End}_R(M)/\text{jac}(\text{End}_R(M))$ is a division ring). No element of $\mathcal{P}_M$ can be an isomorphism, so we have $\mathcal{P}_M \subseteq \text{jac}(\text{End}_R(M))$. For general stable M , write $M = \bigoplus_i M_i$ with each M_i indecomposable nonfree. Then $\text{jac}(\text{End}_R(M))$ is the set of matrices (φ_{ij}) each of whose entries $M_j \rightarrow M_i$ is a non-isomorphism. By the previous case no component of an element of $\mathcal{P}_M$ can be an isomorphism, so again $\mathcal{P}_M \subseteq \text{jac}(\text{End}_R(M))$.

Now $\text{End}_R(M)$ is a module-finite algebra over the complete local ring R , hence is semiperfect, i.e. $\text{End}_R(M)$ is semilocal and idempotents of the quotient $\text{End}_R(M)/\text{jac}(\text{End}_R(M))$ can be lifted to $\text{End}_R(M)$. It follows that the idempotents of $\underline{\text{End}}_R(M) = \text{End}_R(M)/\mathcal{P}_M$ can be lifted as well, finishing the proof. $\square$

Now we begin the proof of Theorem A in earnest. Let $\mathcal{I}$ be the essential image of $(-)^{\star}$ in $D_{\text{sg}}(A)$.

Lemma 4.5. *The subcategory $\mathcal{I}$ is a thick subcategory of $D_{\text{sg}}(A)$: it is closed under isomorphisms, finite direct sums, $\Sigma^{\pm 1}$, cones and direct summands.*

Proof. We have already observed (Theorem 2.9) that $(-)^{\star}$ is a functor of triangulated categories, i.e. commutes with $\Sigma^{\pm 1}$. Given $\Theta: X^{\star} \rightarrow Y^{\star}$, Thm. 2.9 gives $\Theta \cong \theta^{\star}$ for some $\theta: X \rightarrow Y$, and then $\text{cone}(\Theta) \cong \text{cone}(\theta^{\star}) \cong \text{cone}(\theta)^{\star}$ by Prop. 2.7(iii), so $\mathcal{I}$ is closed under cones.

For summands, let X be an object of $D_{\text{sg}}(R)$ and let Y be a summand of $X^{\mathbf{x}}$. Using Buchweitz's equivalence, we may choose a representative for X which is a stable MCM R -module. Since the entries of the matrices in Definition 2.1 are those of φ and ψ together with $\pm u$ and $\pm v$, and all live in the maximal ideal of B , it follows that $X^{\mathbf{x}}$, and hence Y , is stable too. A decomposition $X^{\mathbf{x}} \cong Y \oplus Y'$ determines an idempotent $\tilde{e} \in \underline{\text{End}}_A(X^{\mathbf{x}})$, and by Theorem 2.9 there is a unique $e \in \underline{\text{End}}_R(X)$ with $\tilde{e} = e^{\mathbf{x}}$. Since $(e^2)^{\mathbf{x}} = \tilde{e}^2 = \tilde{e} = e^{\mathbf{x}}$ we get $e^2 = e$. By the proof of Lemma 4.4, $e \in \underline{\text{End}}_R(X)$ lifts to an honest idempotent $\hat{e} \in \text{End}_R(X)$, which determines a decomposition $X = \hat{e}X \oplus (1 - \hat{e})X$ in $\text{MCM}(R)$. Applying $(-)^{\mathbf{x}}$ gives $Y \cong \hat{e}X$. $\square$

Proposition 4.6. *Every object of $D_{\text{sg}}(A)$ lies in $\mathcal{I}$. Equivalently, for every MCM A -module N there is a MCM R -module M with $N \cong M^{\mathbf{x}}$ in $\underline{\text{MCM}}(A)$, and for N stable this is an isomorphism of modules.*

Proof. First we show that every MCM R -module, viewed as an A -module via restriction of scalars along $A \twoheadrightarrow R$, lies in $\mathcal{I}$. Let $M \in \text{MCM}(R)$ and set $N = M^{\mathbf{x}}$. By Lemma 2.4(i) we have

$$N^{bb} = N/(u, v)N \cong M \oplus \text{syzy}_1^R(M).$$

Since u, v is a regular sequence on N , the Koszul complex

$$0 \longrightarrow N \xrightarrow{\begin{bmatrix} -v \\ u \end{bmatrix}} N \oplus N \xrightarrow{[u \ v]} N \longrightarrow 0$$

is quasi-isomorphic to $M \oplus \text{syzy}_1^R M$. Since N lies in $\mathcal{I}$ and $\mathcal{I}$ is closed under cones, the Koszul complex lies in $\mathcal{I}$, and since $\mathcal{I}$ is closed under direct summands we see that M lies in $\mathcal{I}$.

Next, we show that every object Y of $D_{\text{sg}}(A)$ lies in $\mathcal{I}$. By Buchweitz's equivalence, we may choose a representative for Y which is a stable MCM A -module N . By Cor. 3.2 multiplications by u and v are both 0 in $\underline{\text{End}}_A(N)$ and likewise on all the stable Hom groups $\underline{\text{Hom}}_A(\Sigma^i N, \Sigma^j N)$. As above, the Koszul complex on u, v over N is exact, so that in $D_{\text{sg}}(A)$ we have

$$N^{bb} \cong \text{cone}\left(\text{cone}\left(N \xrightarrow{u} N\right) \xrightarrow{v} \text{cone}\left(N \xrightarrow{u} N\right)\right),$$

an iterated cone construction. Both maps are zero in $D_{\text{sg}}(A)$, and a triangle of the form $A \xrightarrow{0} A \longrightarrow C \longrightarrow \Sigma A$ always splits. Hence we have

$$N^{bb} \cong N \oplus (\Sigma N)^2 \oplus \Sigma^2 N$$

in $D_{\text{sg}}(A)$. In particular N is a direct summand of N^{bb} . But N^{bb} is a MCM R -module, hence lives in $\mathcal{I}$ by the previous paragraph, and $\mathcal{I}$ is closed under direct summands, so N lies in $\mathcal{I}$. $\square$

We restate Theorem A for convenience.

Theorem A. *Let $(S, \mathfrak{n}, k)$ be a complete regular local ring, let $0 \neq f \in \mathfrak{n}^2$, and set $R = S/(f)$ and $A = R^{\mathbf{x}} = S[[u, v]]/(f + uv)$. Then the Knörrer functor*

$$(-)^{\mathbf{x}} : \underline{\text{MCM}}(R) \longrightarrow \underline{\text{MCM}}(A)$$

is an equivalence of triangulated categories.

Proof of Theorem A. The triangulated functor $(-)^{\mathbf{x}}$ is fully faithful by Theorem 2.9 and dense by Proposition 4.6. $\square$

The following corollary is [LW12, Thm. 8.33], but without the standing hypotheses there, namely that the residue field k be algebraically closed of characteristic different from 2.

Corollary 4.8. *The functor $(-)^{\mathbf{x}}$ induces a bijection between the isomorphism classes of indecomposable non-free MCM R -modules and those of the indecomposable non-free MCM A -modules.*

Proof. Both $\underline{\text{MCM}}(R)$ and $\underline{\text{MCM}}(A)$ have the Krull-Remak-Schmidt uniqueness property for direct-sum decompositions by [LW12, Cor. 1.10] and Lemma 4.4, so the equivalence $(-)^{\mathbf{x}}$ induces a bijection on the isomorphism classes of the indecomposable objects. $\square$

The next corollary generalizes the main result of [Sol89], without assuming existence of almost split sequences. Combined with Greuel-Kröning [GK90] it gives the “only if” direction of [LW12, Theorem 9.28].

Corollary 4.9. *The hypersurface defined by f has finite (resp., countable) Cohen-Macaulay type if and only if that defined by $f + uv$ does.* $\square$

The next corollary completes Lemma 2.4.

Corollary 4.10. *Let N be a stable MCM A -module. Then*

$$(N^{bb})^{\mathbf{x}} \cong N \oplus \operatorname{syz}_1^A N$$

as A -modules; in particular N is a direct summand of $(N^{bb})^{\mathbf{x}}$.

Proof. By Corollary 4.8 we can write $N \cong M^{\mathbf{x}}$ for some stable MCM R -module M . Then $N^{bb} \cong M \oplus \operatorname{syz}_1^R M$ by Lemma 2.4(i), so

$$(N^{bb})^{\mathbf{x}} \cong (M \oplus \operatorname{syz}_1^R M)^{\mathbf{x}} \cong M^{\mathbf{x}} \oplus (\operatorname{syz}_1^R M)^{\mathbf{x}} \cong M^{\mathbf{x}} \oplus \operatorname{syz}_1^A (M^{\mathbf{x}}) = N \oplus \operatorname{syz}_1^A N,$$

the last isomorphism by Lemma 2.4(ii). Both sides are stable, so the stable isomorphism is an isomorphism of modules. $\square$

Example 4.11. Nothing above requires f to be reduced or irreducible, or to define an isolated singularity. Take $\operatorname{char} k = 2$, $S = k[[x, y]]$ and $f = x^2 y^2$. Then $A = k[[x, y, u, v]]/(x^2 y^2 + uv)$ has singular locus $V(u, v) \cong \operatorname{Spec} k[[x, y]]/(x^2 y^2)$, of dimension one. Theorem A gives an equivalence $\underline{\operatorname{MCM}}(R) \simeq \underline{\operatorname{MCM}}(A)$. This example does not satisfy the hypotheses of [Knö87] ($\operatorname{char} k = 2$) or [Sol89] (not an isolated singularity) or [Dyc11] (fails Condition (B) since not an isolated singularity).

5. KNÖRRER’S FUNCTOR AS A SECOND SYZGY

This section proves Theorem B. The argument is an explicit construction of a free resolution which uses neither Theorem A nor Section 4, and is manifestly characteristic-free.

We keep the notation established in the previous sections, and restate Theorem B for convenience.

Theorem B. *Let M be a MCM R -module with no free direct summands, and regard M as a A -module through $A \twoheadrightarrow A/(u, v) = R$. Then*

$$\operatorname{syz}_2^A(M) \cong M^{\mathbf{x}} \oplus (\operatorname{syz}_1^R M)^{\mathbf{x}} = (M \oplus \operatorname{syz}_1^R M)^{\mathbf{x}},$$

and neither side has a free direct summand.

Proof of Theorem B. Let $(\varphi: G \rightarrow F, \psi: F \rightarrow G)$ be a reduced matrix factorization of f over S with $\operatorname{cok}(\varphi, \psi) = M$, and let n be the size of the matrix φ . We regard M as an A -module via $A \twoheadrightarrow A/(u, v) = R$.

As in [LW12, 8.13], we upgrade the free S -modules F and G to free A -modules $\tilde{F} = A \otimes_S F$ and $\tilde{G} = A \otimes_S G$, each isomorphic to A^n , but to keep the notation cleaner we use the same symbols φ and ψ to denote the induced maps between them.

First we observe that there is an exact sequence

$$\tilde{G} \oplus \tilde{F} \oplus \tilde{F} \xrightarrow{\alpha} \tilde{F} \rightarrow M \rightarrow 0,$$

where $\alpha = [\varphi \ u 1_F \ v 1_F]$. Indeed, since $M \cong F/\varphi(G)$ and is also killed by u and v , we have $M \cong \tilde{F}/(\varphi(\tilde{G}) + u\tilde{F} + v\tilde{F})$. In particular $\operatorname{syz}_2^A M = \ker \alpha$.

Set

$$\beta = \begin{bmatrix} u & \psi & 0 & -v \\ -\varphi & v & v & 0 \\ 0 & 0 & -u & \varphi \end{bmatrix} : \tilde{G} \oplus \tilde{F} \oplus \tilde{F} \oplus \tilde{G} \rightarrow \tilde{G} \oplus \tilde{F} \oplus \tilde{F}.$$

Then $\alpha\beta = 0$ by direct computation. To show that $\ker \alpha \subseteq \operatorname{im} \beta$, let $(W, X, Y) \in \ker \alpha \subseteq \tilde{G} \oplus \tilde{F} \oplus \tilde{F}$ and lift to $\hat{W}, \hat{X}, \hat{Y} \in B^n$, so that

$$\varphi(\hat{W}) + u\hat{X} + v\hat{Y} = gH$$

for some $H \in B^n$. Since $\varphi\psi = f \cdot 1_n$ over S and $g = f + uv$, this gives

$$\varphi(\hat{W}) + u\hat{X} + v\hat{Y} = \varphi\psi(H) + uvH,$$

whence

$$\varphi(\hat{W} - \psi(H)) + u(\hat{X} - vH) + v\hat{Y} = 0$$

in B^n . In other words, $(\hat{W} - \psi(H), \hat{X} - vH, \hat{Y})$ is in the kernel of α when considered as a matrix over B . Identifying the difference between these two lifts as coming from the second column of β , we may assume that (W, X, Y) satisfies

$$\varphi(W) + uX + vY = 0 \tag{5.1}$$

on the nose in B^n , not just in A^n .

Now, setting $u = v = 0$ in (5.1) gives $\varphi(W) = 0$ in $B^n/(u, v)B^n = S^n$. Since φ is injective over S , we have $W = 0$ in S^n , i.e. $W \in (u, v)B^n$. Write $W = uW_1 + vW_2$. Then

$$u(\varphi(W_1) + X) + v(\varphi(W_2) + Y) = 0.$$

Since u, v is a regular sequence on B^n , there exists $Z \in \tilde{F}$ such that

$$\varphi(W_1) + X = vZ \quad \text{and} \quad \varphi(W_2) + Y = -uZ.$$

In particular,

$$(W, X, Y) = (uW_1, -\varphi(W_1), 0) + (0, vZ, -uZ) + (vW_2, 0, -\varphi(W_2))$$

is in the image of the first, third, and fourth columns of β .

Finally consider the matrix

$$\gamma = \begin{bmatrix} \psi & v & v & 0 \\ -u & \varphi & 0 & v \\ 0 & 0 & \varphi & -v \\ 0 & 0 & u & \psi \end{bmatrix} : \tilde{F} \oplus \tilde{G} \oplus \tilde{G} \oplus \tilde{F} \longrightarrow \tilde{G} \oplus \tilde{F} \oplus \tilde{F} \oplus \tilde{G}.$$

Once again $\beta\gamma = 0$ as matrices over A . It follows that $\operatorname{im} \gamma \subseteq \ker \beta$, whence β factors through $\operatorname{cok} \gamma$ and we obtain a surjection $\pi : \operatorname{cok} \gamma \longrightarrow \operatorname{im} \beta = \operatorname{syz}_2^A M$. We will see below that in fact π is an isomorphism, so $\operatorname{im} \gamma = \ker \beta$ and $\operatorname{syz}_2^A M \cong \operatorname{cok} \gamma$. Before proving this fact, we identify $\operatorname{cok} \gamma$.

Note that the two diagonal $2n \times 2n$ blocks of γ are the matrices of Definition 2.1 in reverse order. In particular, if we set $(\Phi, \Psi) = (\varphi, \psi)^\times$, then

$$\gamma = \begin{bmatrix} \Psi & v1 \\ 0 & \Phi \end{bmatrix}.$$

Set $\partial = \frac{\partial}{\partial u}$. Then $\partial(f + uv) = v$, and the second equation of Lemma 3.1, with $h = f + uv$, reads

$$(\partial\Psi)\Phi + \Psi(\partial\Phi) = v1.$$

In particular

$$\begin{bmatrix} 1 & -\partial\Psi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Psi & v1 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} 1 & -\partial\Phi \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \Psi & 0 \\ 0 & \Phi \end{bmatrix}.$$

Checking back to Definition 2.1, we see that

$$\partial_u \Psi = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \partial_u \Phi = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

so this change of basis is defined and invertible over any ring. It follows that

$$\begin{aligned} \operatorname{cok} \gamma &\cong \operatorname{cok} \begin{bmatrix} \psi & v \\ -u & \varphi \end{bmatrix} \oplus \operatorname{cok} \begin{bmatrix} \varphi & -v \\ u & \psi \end{bmatrix} \\ &= \operatorname{cok} \Psi \oplus \operatorname{cok} \Phi \\ &= \operatorname{syz}_1^A(M^\times) \oplus M^\times \\ &= (\operatorname{syz}_1^R M)^\times \oplus M^\times. \end{aligned}$$

In particular it follows that $\operatorname{cok} \gamma$ is a MCM A -module without free direct summands. As an A -module, it has rank $2n$ by Lemma 2.3.

Finally we observe that $\pi: \operatorname{cok} \gamma \rightarrow \operatorname{im} \beta$ is an isomorphism. Indeed, as an A -module M has rank 0 since it is (u, v) -torsion, so from the short exact sequence $0 \rightarrow \operatorname{syz}_1^A M \rightarrow \tilde{F} \rightarrow M \rightarrow 0$ we get $\operatorname{rank} \operatorname{syz}_1^A M = n$, and from $0 \rightarrow \operatorname{syz}_2^A M \rightarrow \tilde{G} \oplus \tilde{F} \oplus \tilde{F} \rightarrow \operatorname{syz}_1^A M \rightarrow 0$ we get $\operatorname{rank} \operatorname{syz}_2^A M = 2n$. Thus π is a surjection between two modules of the same rank $2n$; its kernel must have rank 0 and thus be a torsion module. However $\ker \pi$ lives inside the MCM module $\operatorname{cok} \gamma$, and MCM modules over CM local rings are always torsion-free since the associated primes are a subset of the associated primes of the ring. Thus $\ker \pi = 0$. $\square$

Remark 5.2. Theorem B can be seen as identifying a maximal Cohen-Macaulay approximation. The forward direction of Buchweitz's equivalence $D_{\operatorname{sg}}(A) \simeq \underline{\operatorname{MCM}}(A)$ is induced on a finitely generated module M by writing a CM approximation $0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$, where X is MCM and $\operatorname{pd}_A Y < \infty$. Since Y is perfect, $M \cong X$ in $D_{\operatorname{sg}}(A)$. Over a Gorenstein ring, it is known (the so-called “pitchfork construction” of Auslander-Buchweitz [AB89]) that one may take $X = \Sigma^n \operatorname{syz}_n^A M$ for any $n \geq \operatorname{codepth}_A M$.

For M a MCM R -module, viewed as an A -module through $A \twoheadrightarrow R$, we have $\operatorname{codepth}_A M = 2$ so the CM approximation is $\Sigma^2 \operatorname{syz}_2^A M$. But $\Sigma^2 = \operatorname{id}$ on MCM A -modules as A is a hypersurface. Hence $M \cong \Sigma^2 \operatorname{syz}_2^A M \cong \operatorname{syz}_2^A M$ in $D_{\operatorname{sg}}(A)$, so $\operatorname{syz}_2^A M$ is the stable MCM A -module representing M , and by Theorem B it is $(M \oplus \operatorname{syz}_1^R M)^\times$.

6. EXPLICIT MATRICES AND THE EULER CLASS

Corollary 4.10 asserts that

$$(N^{bb})^\times \cong N \oplus \operatorname{syz}_1^A N$$

for a stable MCM module over the hyperbolic extension A . In terms of matrix factorizations, this solves the following rigidity problem.

Corollary 6.1. *Let (Φ, Ψ) be a reduced matrix factorization of $f + uv$ over B , and let $\varphi = \Phi|_{u=v=0}$, $\psi = \Psi|_{u=v=0}$, so that (φ, ψ) is a matrix factorization of f over S with cokernel N^{bb} . Then the two matrices*

$$\begin{bmatrix} \varphi & -v1 \\ u1 & \psi \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Phi & \\ & \Psi \end{bmatrix}$$

are equivalent over B .

Question 6.2. *Can one express the equivalence of Cor. 6.1 by explicit matrices over B ?*

We first observe that there is no single universal answer to Question 6.2 for all input data.

Proposition 6.3. *Let f, u, v, Φ, Ψ be indeterminates over $\mathbb{Z}$ and let*

$$\mathcal{U} = \mathbb{Z}[f, u, v] \langle \Phi, \Psi \rangle / (\Phi \Psi - (f + uv), \Psi \Phi - (f + uv))$$

be the generic algebra representing the data of a matrix factorization of $f + uv$. Then

$$v \notin \Psi \mathcal{U} + \mathcal{U} \Phi.$$

Consequently there is no universal formula, polynomial in f, u, v, Φ , and Ψ , with integer coefficients, clearing the off-diagonal blocks in $\begin{bmatrix} \varphi & -v1 \\ u1 & \psi \end{bmatrix}$.

Proof. The algebra $\mathcal{U}$ is free over $\mathbb{Z}[f, u, v]$ on the basis $\{1, \Phi, \Psi, \Phi^2, \Psi^2, \dots\}$. The relations on this basis are generated by

$$\begin{aligned} \Phi\Phi^a &= \Phi^{a+1}, & \Psi\Psi^a &= \Psi^{a+1} \\ \Phi\Psi^a &= (f + uv)\Psi^{a-1}, & \Psi\Phi^a &= (f + uv)\Phi^{a-1} \end{aligned}$$

for all $a \geq 1$. It follows that for every element of $\Psi\mathcal{U} + \mathcal{U}\Phi$, the coefficient of the basis element 1 is contained in the ideal $(f + uv)\mathbb{Z}[f, u, v]$. But $v \notin (f + uv)$. $\square$

Lemma 3.1 and the proof of Theorem B hint that the answer should involve homotopies, as we now explain.

Let $N = \text{cok}(\Phi, \Psi)$ be a stable MCM A -module. The proof of Theorem B constructs an A -free resolution of $M = N^{bb}$ starting from

$$\alpha = [\Phi|_{u=v=0} \quad u1 \quad v1].$$

However, one could equally well, in this case, start from $[\Phi \quad u1 \quad v1]$, which has the same cokernel. This simplifies the argument slightly, and eliminates one row of the matrix β , since instead of $\varphi\psi = f \cdot 1 = -uv \cdot 1$ in A , we now have $\Phi\Psi = g \cdot 1 = 0$ in A . We leave the details of the proof to the reader.

Lemma 6.4. *In the notation of §5, we have an exact sequence of free A -modules*

$$\tilde{G} \oplus \tilde{F} \oplus \tilde{F} \oplus \tilde{G} \xrightarrow{\mathcal{A}} \tilde{F} \oplus \tilde{G} \oplus \tilde{G} \oplus \tilde{F} \xrightarrow{d_2} \tilde{G} \oplus \tilde{F} \oplus \tilde{F} \xrightarrow{d_1} \tilde{F} \longrightarrow N^{bb} \longrightarrow 0,$$

where

$$\begin{aligned} d_1 &= [\Phi \quad u1 \quad v1], \\ d_2 &= \begin{bmatrix} \Psi & u & v & 0 \\ 0 & -\Phi & 0 & -v \\ 0 & 0 & -\Phi & u \end{bmatrix}, \\ \mathcal{A} &= \begin{bmatrix} \Phi & u & v & 0 \\ & -\Psi & 0 & -v \\ & & -\Psi & u \\ & & & \Phi \end{bmatrix}. \end{aligned}$$

In particular $\text{syz}_2^A(N^{bb}) \cong \text{cok } \mathcal{A}$. Moreover, $(\mathcal{A}, \mathcal{B})$ is a matrix factorization of $g = f + uv$ over B , where $\mathcal{B}$ is obtained from $\mathcal{A}$ by swapping Φ and Ψ .

The goal now is to clear the off-diagonal blocks of $\mathcal{A}$. Denote

$$t_u = \frac{\partial}{\partial u} \Phi, \quad s_u = \frac{\partial}{\partial u} \Psi, \quad t_v = \frac{\partial}{\partial v} \Phi, \quad s_v = \frac{\partial}{\partial v} \Psi,$$

so that Lemma 3.1 yields homotopies

$$\begin{aligned} \Phi s_v + t_v \Psi &= u1, & \Psi t_u + s_u \Phi &= v1, \\ \Psi t_v + s_v \Phi &= u1, & \Phi s_u + t_u \Psi &= v1. \end{aligned} \tag{6.1}$$

Write $D = \text{diag}(\Phi, -\Psi, -\Psi, \Phi)$, so that $\mathcal{A} = D + U$ with U strictly block upper-triangular and $\text{cok } D \cong N \oplus \text{syz}_1^A N \oplus \text{syz}_1^A N \oplus \text{syz}_2^A N$.

Proposition 6.5. *Set*

$$X = \begin{bmatrix} 0 & t_v & t_u & 0 \\ & 0 & 0 & s_u \\ & & 0 & -s_v \\ & & & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -s_v & -s_u & 0 \\ & 0 & 0 & -t_u \\ & & 0 & t_v \\ & & & 0 \end{bmatrix},$$

both strictly block upper-triangular matrices, so that $1+X$ and $1+Y$ are invertible over B . Then

$$(1+X)\mathcal{A}(1+Y) = D + Z,$$

where Z has all entries equal to zero except for

$$\begin{aligned} Z_{14} &= t_v \Psi t_u - t_u \Psi t_v \\ &= \Phi(s_u t_v - s_v t_u) + E(\Phi), \end{aligned}$$

with $E = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}$.

Proof. The proof is matrix multiplication, using the four equations (6.1) to kill off the four off-diagonal entries of $\mathcal{A}$. The top right corner entry of $(1+X)\mathcal{A}(1+Y)$ is

$$\mathcal{A}_{14} + (XU)_{14} + (UY)_{14} + (XDY)_{14} = 0 + (ut_u - vt_v) + (vt_v - ut_u) + (t_v \Psi t_u - t_u \Psi t_v).$$

For the second expression, use (6.1) to rewrite $t_u \Psi = v - \Phi s_u$ and $t_v \Psi = u - \Phi s_v$ so that

$$Z_{14} = ut_u - vt_v + \Phi(s_u t_v - s_v t_u).$$

□

To proceed, we introduce a $\mathbb{Z}$ -grading on B . Set

$$\deg u = +1, \quad \deg v = -1, \quad \deg S = 0.$$

Then $f + uv$ is homogeneous of degree zero, so there is an induced grading on $A = B/(f + uv)$. The derivation $E = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}$ appearing in Prop. 6.5 is the Euler derivation for this grading, satisfying $E(h) = dh$ for a homogeneous form h of degree d . Note that $E(f + uv) = uv - vu = 0$, so E descends to a derivation on A as well.

Extend the grading to matrices over B or A in the usual way: we say that a matrix Φ is homogeneous if there exist weights c_j on the source basis and r_i on the target basis so that $\deg(\Phi_{ij}) = r_i - c_j$. Note that if (Φ, Ψ) is a matrix factorization of a homogeneous element of B of degree zero, then Φ is homogeneous if and only if Ψ is, with the c_j and r_i swapped.

The next lemma follows from applying homogeneity entrywise.

Lemma 6.6. *Suppose Φ is a matrix, homogeneous with respect to the grading above. Set $W_r = \text{diag}(r_i)$ and $W_c = \text{diag}(c_j)$. Then*

$$E(\Phi) = W_r \Phi - \Phi W_c.$$

For a matrix factorization (φ, ψ) of f over S , the image under the Knörrer functor $\begin{bmatrix} \varphi & -v \\ u & \psi \end{bmatrix}$ is homogeneous with weights $(0, 1)$ on each side. Theorem A says that up to equivalence this is true of every matrix factorization of $f + uv$.

Corollary 6.7. *Every reduced matrix factorization of $f + uv$ over B is equivalent to one homogeneous for the grading given by $\deg S = 0$, $\deg u = +1$, $\deg v = -1$, with weights in $\{0, 1\}$. Consequently, for every stable MCM A -module $N = \text{cok } \Phi$, the class of the Euler derivation*

$$[E(\Phi)] \in \underline{\text{Hom}}_A(\text{syz}_1^A N, N)$$

vanishes.

Proof. By Corollary 4.8, we can write $N \cong M^{\times}$ for some stable MCM R -module M . Then, choosing a reduced matrix factorization (φ, ψ) of f with $\text{cok} \varphi = M$, we get $\Phi \sim \varphi^{\times}$, which is homogeneous with weights $(0^n, 1^n)$ on each side. The second statement is the first together with Lemma 6.6, since $W_r \Phi - \Phi W_c$ is zero in $\underline{\text{Hom}}_A(\text{syz}_1^A N, N)$. $\square$

Theorem 6.8. *Let (Φ, Ψ) be a reduced matrix factorization of $f + uv$ over B , and assume that Φ is homogeneous with weight matrices W_r, W_c in Lemma 6.6. Set $K = s_u t_v - s_v t_u$, and modify the (1,4) entries of the matrices of Proposition 6.5 as below*

$$\tilde{X} = \begin{bmatrix} 0 & t_v & t_u & -W_r \\ & 0 & 0 & s_u \\ & & 0 & -s_v \\ & & & 0 \end{bmatrix}, \tilde{Y} = \begin{bmatrix} 0 & -s_v & -s_u & W_c - K \\ & 0 & 0 & -t_u \\ & & 0 & t_v \\ & & & 0 \end{bmatrix}.$$

Then

$$(1 + \tilde{X})\mathcal{A}(1 + \tilde{Y}) = \text{diag}(\Phi, -\Psi, -\Psi, \Phi)$$

and hence

$$\text{syz}_2^A(N^{bb}) \cong N \oplus \text{syz}_1^A N \oplus \text{syz}_1^A N \oplus N.$$

Proof. The top-right entry of $(1 + \tilde{X})\mathcal{A}(1 + \tilde{Y})$, by Prop. 6.5, is

$$\begin{aligned} Z_{14} + \tilde{X}_{14}\Phi + \Phi\tilde{Y}_{14} &= (E(\Phi) + \Phi K) + (-W_r \Phi) + (\Phi W_c - \Phi K) \\ &= E(\Phi) - W_r \Phi + \Phi W_c \\ &= 0 \end{aligned}$$

by Lemma 6.6. $\square$

Theorem 6.9. *Let (Φ, Ψ) be a reduced matrix factorization of $f + uv$ over B . Then the two matrix factorizations in Cor. 6.1 are equivalent over B .*

Proof. By Corollary 6.7 write $\Phi = Q \varphi^{\times} P$ for some invertible matrices Q, P over B , where (φ, ψ) is a reduced matrix factorization of f over S . Setting $u = v = 0$ gives $\Phi_0 = Q_0 \text{diag}(\varphi, \psi) P_0$ with P_0, Q_0 still invertible, now over S . Thus Φ_0 is equivalent to $\text{diag}(\varphi, \psi)$, and hence $\Phi_0^{\times} \sim \text{diag}(\varphi, \psi)^{\times}$. This last is equivalent by a block permutation to $\text{diag}(\varphi^{\times}, \psi^{\times})$, so $\Phi_0^{\times} \sim \Phi \oplus \Psi$. $\square$

7. AN EXAMPLE

This section gives an example in characteristic 2 to illustrate both Theorems A and B and the failure of the corresponding statements for the intermediate step between R and $R^{\times}$, namely $R^{\sharp}$.

We retain all the notation introduced previously, and add to it the *double branched cover*

$$R^{\sharp} = S[[z]]/(f + z^2).$$

The iterate of this construction, namely

$$R^{\sharp\sharp} = S[[z, w]]/(f + z^2 + w^2)$$

is, when S contains a square root of -1 and 2 is invertible (for example, when k is algebraically closed of characteristic different from 2) isomorphic to $R^{\times} = S[[u, v]]/(f + uv)$ via the change of variable $u = z + iw, v = z - iw$. In characteristic 2 the two rings are not isomorphic.

We have the pair of functors

$$\begin{aligned} (-)^{\sharp}: \text{MCM}(R) &\longrightarrow \text{MCM}(R^{\sharp}), & M^{\sharp} &= \text{syz}_1^{R^{\sharp}} M \\ (-)^{\flat}: \text{MCM}(R^{\sharp}) &\longrightarrow \text{MCM}(R), & N^{\flat} &= N/zN. \end{aligned}$$

These satisfy the following relation.

Proposition 7.1 ([Knö87, Lemma 2.5(i)], [LW12, Prop. 8.18]). *Let N be a stable MCM $R^\sharp$ -module and assume that $\text{char } k \neq 2$. Then*

$$(N^\flat)^\sharp \cong N \oplus \text{syz}_1^{R^\sharp} N. \quad (7.1)$$

The proof of Proposition 7.1 diagonalizes the matrix $\begin{bmatrix} -\varphi & -z \\ z & \varphi \end{bmatrix}$ by a change of basis of determinant 2, which fails in characteristic 2. We will show that the conclusion fails as well.

Example 7.2. Let k be a field of characteristic 2, $S = k[[x]]$, and $f = x^2$, so that $R = k[[x]]/(x^2)$. Then R is a zero-dimensional (A_1) singularity, with exactly two indecomposable MCM (equivalently, finitely generated) modules, namely

$$R = \text{cok}(x^2, 1) \quad \text{and} \quad k = \text{cok}(x, x).$$

The hyperbolic extension is $A = R^\times = k[[x, u, v]]/(x^2 + uv)$. It is another (A_1) singularity, and is an isolated singularity in characteristic 2 as well, since $\text{jac}(x^2 + uv) = (2x, v, u) = (u, v)$.

The double branched cover $R^\sharp = k[[x, z]]/(x^2 + z^2)$ is not an isolated singularity: we have $x^2 + z^2 = (x + z)^2$, so $R^\sharp \cong k[[s, t]]/(s^2)$ is an (A_∞) singularity. It therefore has infinite CM type by Auslander's theorem [LW12, Thm 7.12]; indeed, the MCM modules over the one-dimensional (A_∞) singularity are classified in [BGS87, §4]: there are the two obvious matrix factorizations of size 1, and

$$\left(\begin{bmatrix} s & t^j \\ 0 & s \end{bmatrix}, \begin{bmatrix} s & -t^j \\ 0 & s \end{bmatrix} \right)$$

for every $j \geq 1$. It is important to note that this classification is characteristic-free.

Proposition 7.3. *With the notation of the Example:*

- (i) *A has exactly two indecomposable MCM modules, namely A and $\mathfrak{p} = (u, x)A$. Both lie in the image of $(-)^{\times}$: explicitly $A = R^\times$ and $\mathfrak{p} = k^\times$.*
- (ii) *$(\mathfrak{p}^{bb})^\times \cong \mathfrak{p} \oplus \text{syz}_1^A \mathfrak{p}$ and $\text{syz}_2^A k = \mathfrak{p} \oplus \mathfrak{p} = (k \oplus \text{syz}_1^R k)^\times$, in accordance with Corollary 4.10 and Theorem B.*
- (iii) *(7.1) does not hold. Set $N = R^\sharp/(x + z)R^\sharp$; then N is a stable indecomposable MCM $R^\sharp$ -module with $\text{syz}_1^{R^\sharp} N \cong N$, but $(N^\flat)^\sharp$ is indecomposable. Hence $(N^\flat)^\sharp \not\cong N \oplus \text{syz}_1^{R^\sharp} N$.*

Proof. By Cor. 4.8 the indecomposable non-free MCM A -modules correspond bijectively via $(-)^{\times}$ to the indecomposable non-free R -modules, of which there is only one. So A has exactly two: $A = R^\times$ itself, and

$$k^\times = \text{cok}(x, x)^\times = \text{cok} \begin{bmatrix} x & -v \\ u & x \end{bmatrix}, \quad (7.2)$$

which is identifiable as the height-one prime $\mathfrak{p} = (u, x)$.

Setting $u = v = 0$ in the presentation matrix of (7.2) gives $\text{diag}(x, x)$, so that $\mathfrak{p}^{bb} \cong k \oplus k$. Hence $(\mathfrak{p}^{bb})^\times = \mathfrak{p} \oplus \mathfrak{p}$. Since $\mathfrak{p}$ is two-generated of rank one, the syzygy $\text{syz}_1^A \mathfrak{p}$ also has rank one, and is not A , so $\text{syz}_1^A \mathfrak{p} = \mathfrak{p}$. Back down over R , we have $\text{syz}_1^R k \cong k$ for similar reasons, giving $\text{syz}_2^A k = (k \oplus \text{syz}_1^R k)^\times = \mathfrak{p} \oplus \mathfrak{p}$ as in Theorem B.

Finally consider $N = R^\sharp/(x + z)R^\sharp$ over $R^\sharp \cong k[[x, z]]/(x + z)^2$. As $(x + z, x + z)$ is a reduced matrix factorization, N is a stable MCM $R^\sharp$ -module, indecomposable since the factorization has size 1. We also have $\text{syz}_1^{R^\sharp} N \cong N$ since the factorization is symmetric.

To compute $N^\flat$ set $z = 0$, obtaining $N^\flat = \text{cok}(x, x) \cong k$ over R . Then

$$(N^\flat)^\sharp = \text{cok} \begin{bmatrix} x & -z \\ z & x \end{bmatrix} = \text{cok} \begin{bmatrix} x & z \\ z & x \end{bmatrix}$$

as $-1 = 1$. In the variables $s = x + z$, $t = x$, this becomes

$$\text{cok} \begin{bmatrix} t & s+t \\ s+t & t \end{bmatrix} \cong \text{cok} \begin{bmatrix} t & s+t \\ s & t \end{bmatrix} \cong \text{cok} \begin{bmatrix} t & s \\ s & 0 \end{bmatrix} \cong \text{cok} \begin{bmatrix} s & t \\ 0 & s \end{bmatrix}$$

via elementary row and column operations, all invertible in characteristic 2. By the classification of MCM modules over (A_∞) in [BGS87, §4], the cokernel of this last matrix is indecomposable. On the other hand $N \oplus \operatorname{syz}_1^{R^\dagger} N \cong N \oplus N$ is not. $\square$

REFERENCES

- [AB89] Maurice Auslander and Ragnar-Olaf Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations*, Mém. Soc. Math. France (N.S.) (1989), no. 38, 5–37, Colloque en l’honneur de Pierre Samuel (Orsay, 1987). MR [1044344](#)
- [BGS87] Ragnar-Olaf Buchweitz, Gert-Martin Greuel, and Frank-Olaf Schreyer, *Cohen-Macaulay modules on hypersurface singularities. II*, Invent. Math. **88** (1987), no. 1, 165–182. MR [877011](#)
- [Buc21] Ragnar-Olaf Buchweitz, *Maximal Cohen-Macaulay modules and Tate cohomology*, Mathematical Surveys and Monographs, vol. 262, American Mathematical Society, Providence, RI, 2021, With appendices by Luchezar L. Avramov, Benjamin Briggs, Srikanth B. Iyengar and Janina C. Letz. The manuscript dates from 1986.
- [DL21] Alex S. Dugas and Graham J. Leuschke, *Some extensions of theorems of Knörrer and Herzog-Popescu*, J. Algebra **571** (2021), 94–120. MR [4200711](#)
- [Dyc11] Tobias Dyckerhoff, *Compact generators in categories of matrix factorizations*, Duke Math. J. **159** (2011), no. 2, 223–274. MR [2824483](#)
- [Eis80] David Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 35–64. MR [570778](#)
- [Ese20] Özgür Esentepe, *The cohomology annihilator of a curve singularity*, J. Algebra **541** (2020), 359–379. MR [4017410](#)
- [GK90] G.-M. Greuel and H. Kröning, *Simple singularities in positive characteristic*, Math. Z. **203** (1990), no. 2, 339–354. MR [1033443](#)
- [IT21] Srikanth B. Iyengar and Ryo Takahashi, *The Jacobian ideal of a commutative ring and annihilators of cohomology*, J. Algebra **571** (2021), 280–296. MR [4200721](#)
- [Knö87] Horst Knörrer, *Cohen-Macaulay modules on hypersurface singularities. I*, Invent. Math. **88** (1987), no. 1, 153–164. MR [877010](#)
- [KST07] Hiroshige Kajiura, Kyoji Saito, and Atsushi Takahashi, *Matrix factorizations and representations of quivers II: type ADE case*, Adv. Math. **211** (2007), no. 1, 327–362, Preprint available as [arXiv:math/0511155](#), November 2005. MR [2313537](#)
- [Kun86] Ernst Kunz, *Kähler differentials*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1986. MR [864975](#)
- [LW12] Graham J. Leuschke and Roger Wiegand, *Cohen-Macaulay representations*, Mathematical Surveys and Monographs, vol. 181, American Mathematical Society, Providence, RI, 2012, xviii+367 pages. ISBN: 978-0-8218-7581-0. MR [2919145](#)
- [Miy67] Takehiko Miyata, *Note on direct summands of modules*, J. Math. Kyoto Univ. **7** (1967), 65–69. MR [0214585](#)
- [Mur13] Daniel Murfet, *Residues and duality for singularity categories of isolated Gorenstein singularities*, Compos. Math. **149** (2013), no. 12, 2071–2100. MR [3143706](#)
- [Orl04] Dmitri O. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Tr. Mat. Inst. Steklova **246** (2004), 240–262. MR [2101296](#)
- [Sol89] Øyvind Solberg, *Hypersurface singularities of finite Cohen-Macaulay type*, Proc. London Math. Soc. (3) **58** (1989), no. 2, 258–280. MR [977477](#)
- [Wan94] Hsin-Ju Wang, *On the Fitting ideals in free resolutions*, Michigan Math. J. **41** (1994), no. 3, 587–608. MR [1297711](#)
- [Yos98] Yuji Yoshino, *Tensor products of matrix factorizations*, Nagoya Math. J. **152** (1998), 39–56. MR [1659389](#)

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