

RESEARCH ARTICLE

Bulletin of the London
Mathematical Society

Branched covers and matrix factorizations

Graham J. Leuschke¹  | Tim Tribone² ¹Department of Mathematics, Syracuse
University, Syracuse, New York, USA²Department of Mathematics, University
of Utah, Salt Lake City, Utah, USA

Correspondence

Tim Tribone, Department of
Mathematics, University of Utah, Salt
Lake City, UT 84112, USA.
Email: tim.tribone@utah.edu

Funding information

CUSE; Syracuse University

Abstract

Let $(S, \mathfrak{n})$ be a regular local ring and f a non-zero element of $\mathfrak{n}^2$. A theorem due to Knörrer states that there are finitely many isomorphism classes of maximal Cohen–Macaulay (CM) $R = S/(f)$ -modules if and only if the same is true for the double branched cover of R , that is, the hypersurface ring which is defined by $f + z^2$ in $S[[z]]$. We consider an analogue of this statement in the case of the hypersurface ring defined instead by $f + z^d$ for $d \geq 2$. In particular, we show that this hypersurface, which we refer to as the d -fold branched cover of R , has finite CM representation type if and only if, up to isomorphism, there are only finitely many indecomposable matrix factorizations of f with d factors. As a result, we give a complete list of polynomials f with this property in characteristic zero. Furthermore, we show that reduced d -fold matrix factorizations of f correspond to Ulrich modules over the d -fold branched cover of R .

MSC 2020

13H10, 13C14 (primary)

1 | INTRODUCTION

Let $(S, \mathfrak{n}, \mathbf{k})$ be a complete regular local ring and let $d \geq 2$ be an integer. Assume that $\mathbf{k}$ is algebraically closed and that the characteristic of $\mathbf{k}$ does not divide d . Fix a non-zero element $f \in \mathfrak{n}^2$ and let $R = S/(f)$ be the hypersurface ring defined by f .

A finitely generated module M over a local ring A is called *maximal Cohen–Macaulay* (MCM) if $\text{depth}_A(M) = \dim(A)$, the Krull dimension of A . We will consider MCM modules over R and

over the d -fold branched cover of R , that is, the hypersurface ring $R^\sharp = S[[z]]/(f + z^d)$. The main objective of this paper is to further understand the connection, established for $d = 2$ by Knörrer [10] and extended for $d > 2$ in [13], between MCM $R^\sharp$ -modules and matrix factorizations of f with d factors which we define below.

Definition 1.1. A matrix factorization of f with d factors is a d -tuple of homomorphisms

$$X = (\varphi_1 : F_2 \rightarrow F_1, \varphi_2 : F_3 \rightarrow F_2, \dots, \varphi_d : F_1 \rightarrow F_d)$$

between finitely generated free S -modules of the same rank such that $\varphi_1 \varphi_2 \cdots \varphi_d = f \cdot 1_{F_1}$. We denote the category of matrix factorizations of f with d factors by $\text{MF}_S^d(f)$. If $\text{rank}_S(F_i) = n$ for all i , then we say X is of size n .

A local ring A is said to have *finite Cohen-Macaulay (CM) type* if there are only finitely many isomorphism classes of indecomposable objects in the category $\text{MCM}(A)$ of MCM A -modules. We adopt the following analogous terminology for the representation type of $\text{MF}_S^d(f)$.

Definition 1.2. We say that f has *finite d -MF type* if the category $\text{MF}_S^d(f)$ has, up to isomorphism, only finitely many indecomposable objects.

In [10], Knörrer proved that $R = S/(f)$ has finite CM type if and only if $R^\sharp = S[[z]]/(f + z^2)$ has finite CM type. The correspondence, given by Eisenbud [6, Corollary 6.3], between matrix factorizations and MCM R -modules implies that the number of isomorphism classes of indecomposable objects in $\text{MCM}(R)$ and $\text{MF}_S^2(f)$ differ by only one. Since $R^\sharp$ is also a hypersurface ring, the same is true for $\text{MCM}(R^\sharp)$ and $\text{MF}_{S[[z]]}^2(f + z^2)$. With this in mind, we can state the following version of Knörrer's theorem.

Theorem 1.3 [10, Corollary 2.8]. Let $f \in \mathfrak{n}^2$ be non-zero, $R = S/(f)$, and $d = 2$ so that $R^\sharp = S[[z]]/(f + z^2)$ and $\text{char } \mathbf{k} \neq 2$. Then, the following are equivalent:

- (i) R has finite CM type;
- (ii) f has finite 2-MF type;
- (iii) $R^\sharp$ has finite CM type;
- (iv) $f + z^2$ has finite 2-MF type.

In Section 2, we investigate which of the analogous implications for d -fold factorizations hold when $d \geq 2$. Our main result in this direction is the following.

Theorem A. Let $d \geq 2$. Then, f has finite d -MF type if and only if the d -fold branched cover $R^\sharp = S[[z]]/(f + z^d)$ has finite CM type.

As a consequence of Theorem A, we give a complete list of polynomials which have only finitely many indecomposable d -fold matrix factorizations up to isomorphism.

Theorem B. Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and $S = \mathbf{k}[[y, x_2, \dots, x_r]]$. Assume $0 \neq f \in (y, x_2, \dots, x_r)^2$ and $d > 2$. Then, f has finite d -MF type if and only if, after a possible change of variables, f and d are one of the following:

- (A_1) $y^2 + x_2^2 + \cdots + x_r^2$ for any $d > 2$;
 (A_2) $y^3 + x_2^2 + \cdots + x_r^2$ for $d = 3, 4, 5$;
 (A_3) $y^4 + x_2^2 + \cdots + x_r^2$ for $d = 3$;
 (A_4) $y^5 + x_2^2 + \cdots + x_r^2$ for $d = 3$.

The main tool in Section 2 is a pair of functors $(-)^b : \text{MCM}(R^\sharp) \rightarrow \text{MF}_S^d(f)$ and $(-)^\sharp : \text{MF}_S^d(f) \rightarrow \text{MCM}(R^\sharp)$. The functors we define play a similar role as the ones in [11, Chapter 8, section 2] given by the same notation. Following the results in [11, Chapter 8, section 3] and [10, Proposition 2.7], we investigate the decomposability of $(-)^b$ and $(-)^^\sharp$ in Section 4.

A matrix factorization $X = (\varphi_1, \dots, \varphi_d) \in \text{MF}_S^d(f)$ is called *reduced* if the map φ_k is minimal for each $k \in \mathbb{Z}_d$. Equivalently, X is reduced if, after choosing bases, all entries of φ_k lie in the maximal ideal of S for all $k \in \mathbb{Z}_d$. In the original setting given by Eisenbud ($d = 2$), matrix factorizations which are reduced are the only interesting ones. In particular, the only indecomposable non-reduced matrix factorizations with two factors are $(1, f)$ and $(f, 1)$. For $d > 2$, there are often non-reduced d -fold matrix factorizations of f which are also indecomposable and therefore play a role in Theorem A and Theorem B.

In Section 5, we focus only on reduced matrix factorizations. We show that, as long as d is not too large, reduced matrix factorizations of f with d factors correspond to *Ulrich modules* over $R^\sharp$, that is, modules which are MCM and maximally generated in the sense of [3]. We let $\text{ord}(f)$ denote the maximal integer e such that $f \in \mathfrak{n}^e$.

Theorem C. Assume $d \leq \text{ord}(f)$ and $f + z^d$ is irreducible. Then, f has finite reduced d -MF type if and only if there are, up to isomorphism, only finitely many indecomposable Ulrich $R^\sharp$ -modules.

Notation. All indices are taken modulo d unless otherwise specified.

2 | FINITE MATRIX FACTORIZATION TYPE

The key lemma in Knörrer's proof of Theorem 1.3 states that for MCM modules M over R and N over the double branched cover $R^\sharp = S[[z]]/(f + z^2)$, we have isomorphisms

$$\text{syz}_1^{R^\sharp}(M)/z \text{syz}_1^{R^\sharp}(M) \cong M \oplus \text{syz}_1^R(M)$$

and

$$\text{syz}_1^{R^\sharp}(N/zN) \cong N \oplus \text{syz}_1^{R^\sharp}(N).$$

In this section, we extend Knörrer's lemma to arbitrary d -fold branched covers and use it to prove Theorem A.

As in the Introduction, we fix $d \geq 2$ and let $(S, \mathfrak{n}, \mathbf{k})$ be a complete regular local ring with $\mathbf{k}$ algebraically closed of characteristic not dividing d . We also fix a non-zero element $f \in \mathfrak{n}^2$ and let $R = S/(f)$. We recall a few facts about matrix factorizations of f with two or more factors.

Lemma 2.1 [13]. Let $X = (\varphi_1, \varphi_2, \dots, \varphi_d) \in \text{MF}_S^d(f)$ and $i \in \mathbb{Z}_d$.

- (i) The d -tuple $(\varphi_i, \varphi_{i+1}, \dots, \varphi_d, \varphi_1, \dots, \varphi_{i-1})$ also forms a matrix factorization of f with d factors, that is, $\varphi_i \varphi_{i+1} \cdots \varphi_d \varphi_1 \cdots \varphi_{i-1} = f \cdot 1_{F_i}$.
- (ii) The R -module $\text{cok } \varphi_i$ is MCM.
- (iii) The Krull-Remak-Schmidt Theorem (KRS) holds in the additive category[†] $\text{MF}_S^d(f)$ (since S is complete). That is, any object $X \in \text{MF}_S^d(f)$ decomposes into a finite direct sum of indecomposable objects such that the decomposition is unique up to isomorphism and permutation of the summands.

The d -fold branched cover of R is the hypersurface ring $R^\sharp = S[[z]]/(f + z^d)$. As an S -module, $R^\sharp$ is finitely generated and free with basis given by $\{1, z, z^2, \dots, z^{d-1}\}$. Consequently, a finitely generated $R^\sharp$ -module N is MCM over $R^\sharp$ if and only if it is free over S [14, Proposition 1.9]. Furthermore, multiplication by z on N defines an S -linear map $\varphi : N \rightarrow N$ which satisfies $\varphi^d = -f \cdot 1_N$. Conversely, given a free S -module F and a homomorphism $\varphi : F \rightarrow F$ satisfying $\varphi^d = -f \cdot 1_F$, the pair (F, φ) defines an MCM $R^\sharp$ -module whose z -action is given by the map φ . We will use these perspectives interchangeably throughout.

Since S is complete and $\text{char } \mathbf{k} \nmid d$, [11, A.31] implies that the polynomials $x^d + 1$ and $x^d - 1$ in $S[x]$ each have d distinct roots in S . We let $\omega \in S$ be a *primitive* d th root of 1 in the sense that $\omega^d = 1$ and $\omega^s \neq 1$ for any $1 \leq s < d$. Let $\mu \in S$ be any d th root of -1 . We start with a pair of functors between the categories $\text{MCM}(R^\sharp)$ and $\text{MF}_S^d(f)$.

Definition 2.2.

- (i) Let $N \in \text{MCM}(R^\sharp)$ and set $\varphi : N \rightarrow N$ to be the S -linear homomorphism representing multiplication by z on N . Since $(\mu\varphi)^d = f \cdot 1_N$, we define

$$N^b = (\mu\varphi, \mu\varphi, \dots, \mu\varphi) \in \text{MF}_S^d(f).$$

Given a homomorphism $g : N \rightarrow N'$ of $R^\sharp$ -modules, we define $g^b = (g, g, \dots, g)$ which is a morphism of matrix factorizations in $\text{Hom}_{\text{MF}_S^d(f)}(N^b, (N')^b)$. These operations define a functor $(-)^b : \text{MCM}(R^\sharp) \rightarrow \text{MF}_S^d(f)$.

- (ii) Let $X = (\varphi_1 : F_2 \rightarrow F_1, \varphi_2 : F_3 \rightarrow F_2, \dots, \varphi_d : F_1 \rightarrow F_d) \in \text{MF}_S^d(f)$. Define an MCM $R^\sharp$ -module by setting $X^\sharp = \bigoplus_{k=0}^{d-1} F_{d-k}$ as an S -module with z -action given by:

$$z \cdot (x_d, x_{d-1}, \dots, x_2, x_1) := (\mu^{-1}\varphi_d(x_1), \mu^{-1}\varphi_{d-1}(x_d), \dots, \mu^{-1}\varphi_1(x_2))$$

for all $x_i \in F_i, i \in \mathbb{Z}_d$. Given a morphism $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) : X \rightarrow X'$, define an $R^\sharp$ -homomorphism $\alpha^\sharp = \bigoplus_{k=0}^{d-1} \alpha_{d-k} \in \text{Hom}_{R^\sharp}(X^\sharp, (X')^\sharp)$. This gives a functor $(-)^{\sharp} : \text{MF}_S^d(f) \rightarrow \text{MCM}(R^\sharp)$.

Remark 2.3. The role of μ in the definition of N^b is to obtain a d -fold factorization of f (instead of $-f$) and to do so in a symmetric way. It is important to note that the isomorphism class of $N^b \in \text{MF}_S^d(f)$ is independent of choice of μ . To see this, observe that if μ' is another root of $x^d + 1$, then $\mu' = \omega^j \mu$ for some $j \in \mathbb{Z}_d$. Using this fact, one can construct an isomorphism

[†] Morphisms and direct sums in $\text{MF}_S^d(f)$ are defined component-wise (see [13, 1.4]). The zero object is the matrix factorization with $F_i = 0$ and $\varphi_i = 0$ for all $i \in \mathbb{Z}_d$.

$(\mu\varphi_N, \mu\varphi_N, \dots, \mu\varphi_N) \cong (\mu'\varphi_N, \mu'\varphi_N, \dots, \mu'\varphi_N)$ in $\mathrm{MF}_S^d(f)$. Similarly, μ^{-1} in the definition of $X^\sharp$ ensures that we obtain a module over $R^\sharp$. The isomorphism class of $X^\sharp$ in $\mathrm{MCM}(R^\sharp)$ is also independent of the choice of μ .

Consider the automorphism $\sigma : R^\sharp \rightarrow R^\sharp$ which fixes S pointwise and maps z to ωz . This automorphism acts on the category of $\mathrm{MCM}(R^\sharp)$ -modules in the following sense: For each $N \in \mathrm{MCM}(R^\sharp)$, let $(\sigma^k)^*N$ denote the $\mathrm{MCM}(R^\sharp)$ -module obtained by restricting scalars along σ^k . Since $\sigma^d = 1_{R^\sharp}$, the mapping $N \mapsto \sigma^*N$ forms an autoequivalence of the category $\mathrm{MCM}(R^\sharp)$.

The functor $T : \mathrm{MF}_S^d(f) \rightarrow \mathrm{MF}_S^d(f)$ given by $T(\varphi_1, \varphi_2, \dots, \varphi_d) = (\varphi_2, \varphi_3, \dots, \varphi_d, \varphi_1)$, which we refer to as the *shift functor*, also gives an autoequivalence, this time of $\mathrm{MF}_S^d(f)$.

Proposition 2.4. *Let N be an $\mathrm{MCM}(R^\sharp)$ -module and $X \in \mathrm{MF}_S^d(f)$. Then,*

$$X^\sharp \cong \bigoplus_{k \in \mathbb{Z}_d} T^k(X) \quad \text{and} \quad N^\sharp \cong \bigoplus_{k \in \mathbb{Z}_d} (\sigma^k)^*N.$$

Proof. Let $X = (\varphi_1 : F_2 \rightarrow F_1, \dots, \varphi_d : F_1 \rightarrow F_d) \in \mathrm{MF}_S^d(f)$. Since $X^\sharp = F_d \oplus F_{d-1} \oplus \dots \oplus F_1$ as an S -module, multiplication by z on $X^\sharp$ is given by $\mu^{-1}\varphi$ where

$$\varphi = \begin{pmatrix} 0 & 0 & \dots & 0 & \varphi_d \\ \varphi_{d-1} & 0 & \dots & 0 & 0 \\ 0 & \varphi_{d-2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \varphi_1 & 0 \end{pmatrix}.$$

Therefore, $X^\sharp = (\varphi, \varphi, \dots, \varphi) \in \mathrm{MF}_S^d(f)$. One can perform row and column operations to see that that $(\varphi, \varphi, \dots, \varphi) \cong \bigoplus_{k \in \mathbb{Z}_d} T^k(X)$.

Notice that the first half of the proof is valid in any characteristic as long as there exists an element $\mu \in S$ satisfying $\mu^d = -1$. For instance, if d is odd, then $\mu = -1$ is a valid choice. However, the second half of the proof explicitly makes use of the fact that $\mathrm{char} \mathbf{k}$ does not divide d .

In order to show the second isomorphism, let $N \in \mathrm{MCM}(R^\sharp)$ and set $\varphi : N \rightarrow N$ to be the S -linear map representing multiplication by z on N . Then, $N^\sharp = (\mu\varphi : N \rightarrow N, \dots, \mu\varphi : N \rightarrow N) \in \mathrm{MF}_S^d(f)$. Thus, as an S -module, $N^\sharp = N \oplus N \oplus \dots \oplus N$, the direct sum of d copies of the free S -module N . The z -action on $N^\sharp$ is given by

$$\begin{aligned} z \cdot (n_d, n_{d-1}, \dots, n_1) &= (\mu^{-1}\mu\varphi(n_1), \mu^{-1}\mu\varphi(n_d), \dots, \mu^{-1}\mu\varphi(n_2)) \\ &= (zn_1, zn_d, \dots, zn_2), \end{aligned}$$

for any $n_i \in N, i \in \mathbb{Z}_d$. Let $k \in \mathbb{Z}_d$ and define a map $g_k : N^\sharp \rightarrow (\sigma^k)^*N$ by mapping

$$n = (n_d, n_{d-1}, \dots, n_1) \mapsto \frac{1}{d} \sum_{j=0}^{d-1} \omega^{jk} n_{d-j}$$

for any $n \in N^{b\sharp}$. Note that for $m \in (\sigma^k)^*N$, $z \cdot m = \omega^k z m$ by definition. Therefore, for $n = (n_d, \dots, n_1) \in N^{b\sharp}$, we have that

$$\begin{aligned} z \cdot g_k(n) &= \frac{1}{d} \sum_{j=0}^{d-1} z \cdot (\omega^{jk} n_{d-j}) \\ &= \frac{1}{d} \sum_{j=0}^{d-1} \omega^{(j+1)k} z n_{d-j} \\ &= \frac{1}{d} \left(\omega^k z n_d + \omega^{2k} z n_{d-1} + \dots + \omega^{(d-1)k} z n_2 + z n_1 \right) \\ &= g_k(z \cdot n). \end{aligned}$$

In other words, g_k is an $R^\sharp$ -homomorphism. Putting these maps together we have an $R^\sharp$ -homomorphism

$$g = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{d-1} \end{pmatrix} : N^{b\sharp} \rightarrow \bigoplus_{k=0}^{d-1} (\sigma^k)^* N.$$

In the other direction, we have $R^\sharp$ -homomorphisms $s_k : (\sigma^k)^* N \rightarrow N^{b\sharp}$, $k \in \mathbb{Z}_d$, given by

$$s_k(m) = (m, \omega^{-k} m, \omega^{-2k} m, \dots, \omega^{-(d-1)k} m)$$

for any $m \in (\sigma^k)^* N$. For each $k \in \mathbb{Z}_d$ and $m \in (\sigma^k)^* N$, we have $g_k s_k(m) = m$. On the other hand, if $i \neq \ell \in \mathbb{Z}_d$, then

$$\begin{aligned} g_i s_\ell(m) &= g_i(m, \omega^{-\ell} m, \omega^{-2\ell} m, \dots, \omega^\ell m) \\ &= \frac{1}{d} \sum_{j=0}^{d-1} \omega^{j(i-\ell)} m \\ &= 0. \end{aligned}$$

Therefore, setting $s = (s_0 \ s_1 \ \dots \ s_{d-1})$, we have that gs is the identity on $\bigoplus_{k=0}^{d-1} (\sigma^k)^* N$ and so g is a split surjection. However, since both the target and source of g have the same rank as free S -modules, we can conclude that g is an isomorphism of $R^\sharp$ -modules. $\square$

Corollary 2.5.

- (i) For each $X \in MF_S^d(f)$, there exists $N \in MCM(R^\sharp)$ such that X is isomorphic to a summand of $N^{b\sharp}$.
- (ii) For each $N \in MCM(R^\sharp)$, there exists $X \in MF_S^d(f)$ such that N is isomorphic to a summand of $X^\sharp$.

We can now prove Theorem A. The proof is lifted directly from the $d = 2$ case. Once again, the characteristic assumption on $\text{char } \mathbf{k}$ is only needed in half of the proof as long as there exists $\mu \in S$ satisfying $\mu^d = -1$.

Proof of Theorem A. Let $X_1, X_2, \dots, X_t$ be a representative list of the isomorphism classes of indecomposable d -fold matrix factorizations of f and let $N \in \text{MCM}(R^\sharp)$ be indecomposable. Since $N^\flat \in \text{MF}_S^d(f)$, there exist non-negative integers $s_1, s_2, \dots, s_t$ such that $N^\flat \cong X_1^{s_1} \oplus X_2^{s_2} \oplus \dots \oplus X_t^{s_t}$. By Proposition 2.4, N is isomorphic to a direct summand of

$$N^{\flat\sharp} \cong (X_1^\sharp)^{s_1} \oplus (X_2^\sharp)^{s_2} \oplus \dots \oplus (X_t^\sharp)^{s_t}.$$

Since N is indecomposable, KRS in $\text{MCM}(R^\sharp)$ implies that N is isomorphic to a summand of $X_i^\sharp$ for some $1 \leq i \leq t$. Hence, every indecomposable $\text{MCM}(R^\sharp)$ -module is isomorphic to one appearing in the finite list consisting of all summands of all $X_j^\sharp$, $1 \leq j \leq t$. The converse follows similarly from Proposition 2.4 and KRS in $\text{MF}_S^d(f)$. $\square$

As above, $\sigma : R^\sharp \rightarrow R^\sharp$ is the automorphism of $R^\sharp$ which pointwise fixes S and maps z to ωz . In [13, Section 4], it was shown that the category $\text{MF}_S^d(f)$ is equivalent to the category of finitely generated modules over the skew group algebra $R^\sharp[\sigma]$ which are MCM as $R^\sharp$ -modules (equivalently, free as S -modules). We denote this category by $\text{MCM}_\sigma(R^\sharp)$. The equivalence is given by a pair of inverse functors $\mathcal{A} : \text{MCM}_\sigma(R^\sharp) \rightarrow \text{MF}_S^d(f)$ and $\mathcal{B} : \text{MF}_S^d(f) \rightarrow \text{MCM}_\sigma(R^\sharp)$ defined as follows: For $X = (\varphi_1 : F_2 \rightarrow F_1, \dots, \varphi_d : F_1 \rightarrow F_d) \in \text{MF}_S^d(f)$, $\mathcal{B}(X) := X^\sharp$ as an $R^\sharp$ -module with the action of σ given by

$$\sigma \cdot (x_d, x_{d-1}, \dots, x_1) = (x_d, \omega x_{d-1}, \dots, \omega^{d-1} x_1)$$

for all $x_i \in F_i, i \in \mathbb{Z}_d$. For any $N \in \text{MCM}_\sigma(R^\sharp)$, N decomposes as an S -module into $N \cong \bigoplus_{i \in \mathbb{Z}_d} N^{\omega^i}$ [13, Lemma 4.2] where N^{ω^i} is the σ -invariant subspace $N^{\omega^i} = \{x \in N : \sigma(x) = \omega^i x\}$. For each $i \in \mathbb{Z}_d$, $zN^{\omega^i} \subseteq N^{\omega^{i+1}}$ and therefore multiplication by z defines an S -linear map $N^{\omega^i} \rightarrow N^{\omega^{i+1}}$. Since $(\mu z)^d = f$, the composition

$$N^{\omega^{d-1}} \xrightarrow{\mu z} N^1 \xrightarrow{\mu z} N^\omega \xrightarrow{\mu z} \dots \xrightarrow{\mu z} N^{\omega^{d-2}} \xrightarrow{\mu z} N^{\omega^{d-1}}$$

defines a matrix factorization which we denote as $\mathcal{A}(N) \in \text{MF}_S^d(f)$.

To finish this section, we make note of the connection between the functors $\mathcal{A}$ and $\mathcal{B}$ and the functors $(-)^{\sharp}$ and $(-)^{\flat}$.

Lemma 2.6. *Let $H : \text{MCM}_\sigma(R^\sharp) \rightarrow \text{MCM}(R^\sharp)$ be the functor which forgets the action of σ and $G : \text{MCM}(R^\sharp) \rightarrow \text{MCM}_\sigma(R^\sharp)$ be given by $G(N) = R^\sharp[\sigma] \otimes_{R^\sharp} N$ for any $N \in \text{MCM}(R^\sharp)$.*

- (i) *For any $X \in \text{MF}_S^d(f)$, $X^\sharp = H \circ \mathcal{B}(X)$.*
- (ii) *For any $N \in \text{MCM}(R^\sharp)$, $N^\flat \cong \mathcal{A} \circ G(N)$.*

Proof. The first statement follows directly from the definition of $(-)^{\sharp}$ and $\mathcal{B}$. For the second, consider the idempotents

$$e_k = \frac{1}{d} \sum_{j \in \mathbb{Z}_d} \omega^{-jk} \sigma^j \in R^\sharp[\sigma], \quad k \in \mathbb{Z}_d.$$

These idempotents have three important properties:

- (a) $R^\sharp[\sigma] = \bigoplus_{k \in \mathbb{Z}_d} e_k R^\sharp[\sigma]$ as right $R^\sharp[\sigma]$ -modules,
- (b) $\sigma e_k = e_k \sigma = \omega^k e_k$, $k \in \mathbb{Z}_d$, and
- (c) $ze_k = e_{k-1} z$, $k \in \mathbb{Z}_d$.

From (b), we have $e_k R^\sharp[\sigma] = e_k R^\sharp$, where $e_k R^\sharp$ denotes the multiples of e_k by $R^\sharp \cdot 1 \subset R^\sharp[\sigma]$ on the right. Hence, as an $R^\sharp$ -module, $e_k R^\sharp[\sigma] = e_k R^\sharp$ is free of rank 1. Thus, for any $N \in \text{MCM}(R^\sharp)$, (a) implies $G(N) \cong \bigoplus_{k \in \mathbb{Z}_d} (e_k R^\sharp \otimes_{R^\sharp} N)$. It then follows from (b) and (c) that we have an isomorphism of $R^\sharp[\sigma]$ -modules $G(N) \cong B(N^b)$. Hence, $\mathcal{A} \circ G(N) \cong \mathcal{A} \circ B(N^b) \cong N^b$ by [13, Theorem 4.4]. $\square$

Remark 2.7. In the case of an Artin algebra Λ , the relationship between Λ and the skew group algebra $\Lambda[G]$ for a finite group G was studied by Reiten and Riedtmann in [12]. They show that many representation theoretic properties hold simultaneously for Λ and $\Lambda[G]$. In particular, Λ has finite representation type if and only if the same is true of $\Lambda[G]$. The equivalence of categories $\text{MF}_S^d(f) \approx \text{MCM}_\sigma(R^\sharp)$ [13, Theorem 4.4] and Theorem A give an analogous relationship between $R^\sharp$ and the skew group algebra $R^\sharp[\sigma]$.

3 | HYPERSURFACES OF FINITE MATRIX FACTORIZATION TYPE

Let $(A, \mathfrak{m})$ be a regular local ring and $g \in \mathfrak{m}^2$ be non-zero. Then, the hypersurface ring $A/(g)$ is called a *simple hypersurface singularity* if there are only finitely many proper ideals $I \subset A$ such that $g \in I^2$. In the case that A is a power series ring over an algebraically closed field of characteristic zero, the pair of papers [4] and [10] prove the following theorem.

Theorem 3.1 [4, 10]. *Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and let $R = \mathbf{k}[[x_1, x_2, \dots, x_r]]/(g)$, where $g \in (x_1, x_2, \dots, x_r)^2$ is non-zero. Then, R has finite CM type if and only if R is a simple hypersurface singularity.*

Essential to their conclusion is the classification of simple hypersurface singularities, due to Arnol'd [1], which gives explicit normal forms for all polynomials defining such a singularity. These are often referred to as the ADE singularities. The culmination of these results is a complete list of polynomials which define hypersurface rings of finite CM type in all dimensions (see [14, Theorem 8.8] or [11, Theorem 9.8]). Equivalently, the polynomials in this list are precisely the ones with only finitely many indecomposable two-fold matrix factorizations up to isomorphism.

Using Theorem A and the classification described above, we are able to compile a list of all f with finite d -MF type for $d > 2$.

Theorem 3.2. *Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and $S = \mathbf{k}[[y, x_2, \dots, x_r]]$. Assume $0 \neq f \in (y, x_2, \dots, x_r)^2$ and $d > 2$. Then, f has finite d -MF type if and only if, after a possible change of variables, f and d are one of the following:*

- $(A_1): y^2 + x_2^2 + \cdots + x_r^2$ for any $d > 2$;
 $(A_2): y^3 + x_2^2 + \cdots + x_r^2$ for $d = 3, 4, 5$;
 $(A_3): y^4 + x_2^2 + \cdots + x_r^2$ for $d = 3$;
 $(A_4): y^5 + x_2^2 + \cdots + x_r^2$ for $d = 3$.

Proof. Let f and d be a pair in the list given. Then, $f + z^d \in S[[z]]$ defines a simple hypersurface singularity and therefore $R^\sharp$ has finite CM type by Theorem 3.1. By Theorem A, f has finite d -MF type.

Conversely, let $0 \neq f \in (y, x_2, \dots, x_r)^2$, $d > 2$, and assume f has finite d -MF type. Then, $R^\sharp = S[[z]]/(f + z^d)$ has dimension r and is of finite CM type by Theorem A. We consider two cases.

First, assume $\dim R^\sharp = 1$, that is, assume $S = \mathbf{k}[[y]]$. Then, $f = uy^k$ for some unit $u \in S$ and $k \geq 2$. Since S is complete and $\text{char } \mathbf{k} = 0$, there exists a k th root v of u^{-1} in S [11, A.31]. Therefore, after replacing y with vy , we may assume $f = y^k$. Since $\dim R^\sharp = 1$, [14, 8.2.1] implies $\text{ord}(y^k + z^d) \leq 3$. Hence, either $k \leq 3$ or $d \leq 3$. If $k = 2$, there are no restrictions on d since $y^2 + z^d$ defines a simple (A_{d-1}) singularity for all $d > 2$. If $k = 3$, then the fact that $y^3 + z^d$ defines a one-dimensional simple hypersurface singularity implies $d = 3, 4$, or 5 . Similarly, if $d = 3$, then $k = 2, 3, 4$, or 5 .

Next, assume $\dim R^\sharp \geq 2$. In this case, [14, 8.2.2] implies $\text{ord}(f + z^d) \leq 2$. Since $d > 2$ and $f \in (y, x_2, \dots, x_r)^2$, we have $\text{ord}(f) = 2$. By the Weierstrass Preparation Theorem [11, Corollary 9.6], there exists a unit $u \in S$ and $g \in \mathbf{k}[[y, x_2, \dots, x_{r-1}]]$ such that $f = (g + x_r^2)u$. As above, we may neglect the unit and assume $f = g + x_r^2$ for some $g \in \mathbf{k}[[y, x_2, \dots, x_{r-1}]]$.

Since $f + z^d = g + x_r^2 + z^d$ has finite 2-MF type, Knörrer's theorem (Theorem 1.3) implies that $g + z^d$ has finite 2-MF type as well. Thus, g has finite d -MF type by Theorem A. We repeat this argument until $f = g' + x_2^2 + \cdots + x_r^2$ for some $g' \in \mathbf{k}[[y]]$ with finite d -MF type. Finally, we apply the first case to g' to finish the proof. $\square$

Corollary 3.3. *Let $\mathbf{k}$ be an algebraically closed field of characteristic zero, $S = \mathbf{k}[[y, x_2, \dots, x_r]]$, and $f \in (y, x_2, \dots, x_r)^2$ be non-zero. If f has finite d -MF type for some $d \geq 2$, then $R = S/(f)$ is an isolated singularity, that is, $R_{\mathfrak{p}}$ is a regular local ring for all non-maximal prime ideals $\mathfrak{p}$.*

Proof. The polynomials listed in Theorem 3.2 are a subset of the ones in [14, Theorem 8.8] (or [11, Theorem 9.8]), all of which define isolated singularities. $\square$

Suppose we have a pair f and d from the list in Theorem 3.2 such that $R^\sharp$ has dimension 1. Then, [14, Chapter 9] gives matrix factorizations for every indecomposable MCM $R^\sharp$ -module. By computing multiplication by z on each of these $R^\sharp$ -modules, we can compile a representative list of all isomorphism classes of indecomposable d -fold factorizations of f . We give one such computation in the following example.

Example 3.4. Let $\mathbf{k}$ be algebraically closed of characteristic zero. Let $S = \mathbf{k}[[y]]$, $f = y^4 \in S$, and $R = S/(f)$. The hypersurface ring $R^\sharp = \mathbf{k}[[x, y]]/(y^4 + x^3)$ is a simple curve singularity of type E_6 and has finite CM type. Here, we are viewing $R^\sharp$ as the three-fold branched cover of R . By Theorem A, the category $\text{MF}_S^3(y^4)$ has only finitely many non-isomorphic indecomposable objects. We give a complete list below.

A complete list of non-isomorphic indecomposable MCM $R^\sharp$ -modules is given in [14, 9.13]. By Corollary 2.5, we may compute multiplication by x on each of these modules to obtain a

representative from each isomorphism class of indecomposable matrix factorizations of y^4 with three factors. By Remark 2.3, we may choose $\mu = -1$.

Following the notation of [14, 9.13], we let $\varphi_1 = \begin{pmatrix} x & y \\ y^3 & -x^2 \end{pmatrix}$ and $M_1 = \text{cok } \varphi_1$. Let e_1 and e_2 in M_1 denote the images of the standard basis on $S[[x]]^2$. Then, e_1 and e_2 satisfy $xe_1 = -y^3e_2$ and $x^2e_2 = ye_1$. As an S -module, M_1 is free with basis $\{e_1, e_2, xe_2\}$. Multiplication by x on M_1 is therefore given by

$$\varphi = \begin{pmatrix} 0 & 0 & y \\ -y^3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence, $M_1^b = (-\varphi, -\varphi, -\varphi) \in \text{MF}_S^3(y^4)$. Furthermore, we have a commutative diagram

$$\begin{array}{ccccccc} S^3 & \xrightarrow{-\varphi} & S^3 & \xrightarrow{-\varphi} & S^3 & \xrightarrow{-\varphi} & S^3 \\ \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right) \downarrow & & \left(\begin{smallmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) \downarrow & & \downarrow \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right) & & \downarrow \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right) \\ S^3 & \xrightarrow{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & y^3 & 0 \\ 0 & 0 & y \end{smallmatrix}\right)} & S^3 & \xrightarrow{\left(\begin{smallmatrix} y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y^3 \end{smallmatrix}\right)} & S^3 & \xrightarrow{\left(\begin{smallmatrix} y^3 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} & S^3. \end{array}$$

Thus, M_1^b is isomorphic to the direct sum of the indecomposable factorization $X_{\varphi_1} := (y^3, y, 1)$ and its corresponding shifts, that is, $M_1^b \cong \bigoplus_{i \in \mathbb{Z}_3} T^i(y^3, y, 1)$.

Similarly, multiplication by x can be computed for each of the indecomposable MCM $R^\sharp$ modules listed in [14, 9.13]. From this computation, we obtain a list of three-fold matrix factorizations of y^4 given in the table below.

$X \in \text{MF}_S^3(y^4)$	N^b for $N \in \text{MCM}(R^\sharp)$
$\mathcal{P}_1 = (y^4, 1, 1)$	$(R^\sharp)^b \cong \bigoplus_{i \in \mathbb{Z}_3} T^i(\mathcal{P}_1)$
$X_{\varphi_1} = (y^3, y, 1)$	$M_1^b \cong \bigoplus_{i \in \mathbb{Z}_3} T^i(X_{\varphi_1})$
$X_{\psi_1} = (y^3, 1, y)$	$N_1^b \cong \bigoplus_{i \in \mathbb{Z}_3} T^i(X_{\psi_1})$
$X_{\varphi_2} = (y^2, y^2, 1)$	$M_2^b \cong N_2^b \cong \bigoplus_{i \in \mathbb{Z}_3} T^i(X_{\varphi_2})$
$X_\beta = (y^2, y, y)$	$B^b \cong \bigoplus_{i \in \mathbb{Z}_3} T^i(X_\beta)$
$X_\alpha = \left(\begin{pmatrix} 0 & -y^2 \\ 1 & -y \end{pmatrix}, \begin{pmatrix} 0 & -y^3 \\ 1 & -y^2 \end{pmatrix}, \begin{pmatrix} 0 & -y^3 \\ 1 & -y \end{pmatrix} \right)$	$A^b \cong \bigoplus_{i \in \mathbb{Z}_3} T^i(X_\alpha)$
$X_\xi = \left(\begin{pmatrix} y & 0 \\ 0 & y^3 \end{pmatrix}, \begin{pmatrix} 0 & y \\ y & 1 \end{pmatrix}, \begin{pmatrix} -y & 1 \\ y^2 & 0 \end{pmatrix} \right)$	$X^b \cong \bigoplus_{i \in \mathbb{Z}_3} T^i(X_\xi).$

The factorizations $\mathcal{P}_1, X_{\varphi_1}, X_{\psi_1}, X_{\varphi_2}$, and X_β are each indecomposable since they are of size 1. By [13, Corollary 5.15], the syzygy (and cosyzygy) of an indecomposable reduced matrix factorization is again indecomposable. Here, a reduced matrix factorization means all the entries of all the matrices lie in the maximal ideal of S (see Section 5). Using [13, 2.11], we have that $\Omega_{\text{MF}_S^3(y^4)}^-(X_\beta) \cong X_\alpha$ and therefore, X_α is indecomposable.

Since X_ξ is of size 2, a non-trivial decomposition would be of the form $(y, b, c) \oplus (y^3, b', c')$ for some $b, c, b', c' \in S$. Since $\det \begin{pmatrix} 0 & y \\ y & 1 \end{pmatrix} = -y^2$, the possibilities for b and b' are, up to units, $b = y^2$

and $b' = 1$ or $b = y = b'$. By considering cokernels, both cases lead to contradictions and so X_ξ must be indecomposable.

By Proposition 2.4, these seven factorizations, and each of their corresponding shifts, give the complete list of non-isomorphic indecomposable objects in $\text{MF}_S^3(y^4)$ (21 in total).

To end this section, we discuss the relationship between the 2-MF type of f and the d -MF type of f for $d > 2$. In one direction, we have the following consequence of Theorem B.

Corollary 3.5. *Let S be a regular local ring, f a non-zero non-unit in S , and $d \geq 2$. If f has finite d -MF type, then f has finite k -MF type for all $2 \leq k \leq d$. In particular, if f has finite d -MF type for some $d \geq 2$, then $R = S/(f)$ has finite CM type.*

In general, the converse of Lemma 3.5 does not hold. The example below gives a polynomial of finite 2-MF type but of infinite 3-MF type.

Example 3.6. Let $S = \mathbf{k}[[x, y]]$ for an algebraically closed field with characteristic $\text{char } \mathbf{k} \neq 2, 3$ and let $f = x^3 + y^3 \in S$. The hypersurface ring $R = S/(f)$ is a simple singularity of type D_4 and therefore has finite CM type.

Consider $R^\sharp = \mathbf{k}[[x, y, z]]/(x^3 + y^3 + z^3)$, the three-fold branched cover of R . Following [5], to each point $(a, b, c) \in \mathbf{k}^3$ satisfying $a^3 + b^3 + c^3 = 0$ and $abc \neq 0$, we associate the Moore matrix

$$X_{abc} = \begin{pmatrix} ax & bz & cy \\ by & cx & az \\ cz & ay & bx \end{pmatrix}.$$

The $R^\sharp$ -module $N_{abc} := \text{cok}(X_{abc})$ is MCM and is given by the matrix factorization $(X_{abc}, \frac{1}{abc} \text{adj} X_{abc}) \in \text{MF}_{S[[z]]}^2(x^3 + y^3 + z^3)$, where $\text{adj} X_{abc}$ is the classical adjoint of X_{abc} . Furthermore, N_{abc} is indecomposable since $\det X_{abc} = abc(x^3 + y^3 + z^3)$ and $x^3 + y^3 + z^3$ is irreducible. Buchweitz and Pavlov give precise conditions for X_{abc} to be matrix equivalent to $X_{a'b'c'}$ (see [5, Proposition 2.13]). In particular, their results imply that the collection $\{N_{abc}\}$, as (a, b, c) varies over the curve $x^3 + y^3 + z^3$, gives an infinite collection of non-isomorphic indecomposable MCM $R^\sharp$ -modules. It now follows from Theorem A that $x^3 + y^3$ has infinite 3-MF type.

With respect to the images of the standard basis on $S[[z]]^3$, multiplication by z on N_{abc} is given by the S -matrix

$$\varphi_{abc} = \begin{pmatrix} 0 & -\frac{c}{a}y & -\frac{a}{c}x \\ -\frac{c}{b}x & 0 & -\frac{b}{c}y \\ -\frac{a}{b}y & -\frac{b}{a}x & 0 \end{pmatrix}.$$

Therefore, we have $N_{abc}^\flat = (\mu\varphi_{abc}, \mu\varphi_{abc}, \mu\varphi_{abc}) \in \text{MF}_S^3(x^3 + y^3)$, where $\mu^3 = -1$. Hence, the collection of non-isomorphic indecomposable summands of $N_{abc}^\flat$, for all (a, b, c) as above, forms an infinite collection of indecomposable objects in $\text{MF}_S^d(f)$. Furthermore, the entries of φ_{abc} lie in the maximal ideal of S so $x^3 + y^3$ has infinite reduced 3-MF type as well (see Section 5).

4 | DECOMPOSABILITY OF N^b AND $X^\sharp$

Let $d \geq 2$ and $(S, \mathfrak{n}, \mathbf{k})$ be a complete regular local ring. We maintain the same assumptions on $\mathbf{k}$ as in Section 2, that is, we assume that $\mathbf{k}$ is algebraically closed of characteristic not dividing d . Let $f \in \mathfrak{n}^2$ be non-zero, $R = S/(f)$, and $R^\sharp = S[[z]]/(f + z^d)$.

Proposition 2.4 showed that both N^b and $X^\sharp$ decompose into a sum of d objects. In this section, we investigate the decomposability of N^b and $X^\sharp$.

Recall that the shift functor $T : \mathrm{MF}_S^d(f) \rightarrow \mathrm{MF}_S^d(f)$ satisfies $T^d = 1_{\mathrm{MF}_S^d(f)}$. In particular, for any $X \in \mathrm{MF}_S^d(f)$, there exists a smallest integer $k \in \{1, 2, \dots, d-1, d\}$ such that $T^k X \cong X$. We call k the *order* of X .

Lemma 4.1. *For any $X \in \mathrm{MF}_S^d(f)$, the order of X is a divisor of d .*

Proof. For a given $X \in \mathrm{MF}_S^d(f)$, the cyclic group of order d generated by T acts on the set of equivalence classes $\{[T^i X] : i \in \mathbb{Z}_d\}$. In particular, the stabilizer of $[X]$ is generated by T^k for some $k \mid d$ which can be taken to be the smallest possible in $\{1, 2, \dots, d\}$. It follows that the order of X is k . $\square$

The next result builds on an idea of Knörrer [10, Lemma 1.3] and Gabriel [7, p. 95]. The proof is based on [11, Lemma 8.25] which states that a matrix factorization $(\varphi, \psi) \in \mathrm{MF}_S^2(f)$ satisfying $(\varphi, \psi) \cong (\psi, \varphi)$ is isomorphic to a factorization of the form $(\varphi_0, \varphi_0)^\dagger$. For $d > 2$, the situation is similar, but the divisors of d play a role. Specifically, if X has order k , then X is isomorphic to the concatenation of k matrices, repeated d/k times.

Proposition 4.2. *Let $X \in \mathrm{MF}_S^d(f)$ be indecomposable of size n and assume X has order $k < d$. Then, there exist S -homomorphisms $\varphi'_1, \varphi'_2, \dots, \varphi'_k$ such that $(\varphi'_1 \varphi'_2 \dots \varphi'_k)^{\frac{d}{k}} = f \cdot I_n$ and*

$$X \cong (\varphi'_1, \varphi'_2, \dots, \varphi'_k, \varphi'_1, \varphi'_2, \dots, \varphi'_k, \dots, \varphi'_1, \varphi'_2, \dots, \varphi'_k).$$

Proof. Let $X = (\varphi_1 : F_2 \rightarrow F_1, \varphi_2 : F_3 \rightarrow F_2, \dots, \varphi_d : F_1 \rightarrow F_d)$ and set $r = d/k$. By assumption, there is an isomorphism $\alpha = (\alpha_1, \dots, \alpha_d) : X \rightarrow T^k X$. By applying $T^k(-)$ repeatedly, we obtain an automorphism $\tilde{\alpha}$ of X defined by the composition

$$X \xrightarrow{\alpha} T^k X \xrightarrow{T^k(\alpha)} T^{2k} X \xrightarrow{T^{2k}(\alpha)} \dots \xrightarrow{T^{(r-2)k}(\alpha)} T^{(r-1)k} X \xrightarrow{T^{(r-1)k}(\alpha)} X.$$

In particular, $\tilde{\alpha}$ is the d -tuple $(\alpha_{i+(r-1)k} \alpha_{i+(r-2)k} \dots \alpha_{i+k} \alpha_i)_{i=1}^d$. Since X is indecomposable, the endomorphism ring $\Lambda := \mathrm{End}_{\mathrm{MF}_S^d(f)}(X)$ is local. Since $\mathbf{k}$ is algebraically closed, it cannot have any non-trivial finite extensions which are division rings. Hence the division ring $\Lambda/\mathrm{rad} \Lambda$ must be isomorphic to $\mathbf{k}$. This allows us to write $\tilde{\alpha} = c \cdot 1_X + \rho$ for some $c \in \mathbf{k}^\times$ and $\rho \in \mathrm{rad} \Lambda$. Since $\mathrm{char} \mathbf{k} \nmid d$, we may scale α by $c^{-\frac{1}{r}}$ and assume $\tilde{\alpha} = 1_X + \rho$ for $\rho = (\rho_1, \rho_2, \dots, \rho_d) \in \mathrm{rad} \Lambda$.

[†] The proof of [11, Lemma 8.25] contains a small typo which we remedy in the proof of Proposition 4.2. Specifically, the morphism (α, β) should be replaced by $(c^{-1/2}\alpha, c^{-1/2}\beta)$.

If $i \in \mathbb{Z}_d$, then

$$\begin{aligned}\alpha_i \rho_i &= \alpha_i (\alpha_{i+(r-1)k} \alpha_{i+(r-2)k} \cdots \alpha_{i+k} \alpha_i - 1_{F_i}) \\ &= (\alpha_i \alpha_{i+(r-1)k} \alpha_{i+(r-2)k} \cdots \alpha_{i+k} - 1_{F_{i+k}}) \alpha_i \\ &= \rho_{i+k} \alpha_i.\end{aligned}$$

Represent the function $g(x) = (1+x)^{-1/r}$ by its Maclaurin series and define, for each $i \in \mathbb{Z}_d$,

$$\beta_i := \alpha_i g(\rho_i) = g(\rho_{i+k}) \alpha_i : F_i \rightarrow F_{i+k}.$$

For $i \in \mathbb{Z}_d$, we have

$$\beta_i \varphi_i = g(\rho_{i+k}) \alpha_i \varphi_i = g(\rho_{i+k}) \varphi_{i+k} \alpha_{i+1} = \varphi_{i+k} g(\rho_{i+k+1}) \alpha_{i+1} = \varphi_{i+k} \beta_{i+1}.$$

Hence, $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \text{Hom}_{\text{MF}_{S(f)}^d}(X, T^k X)$. By repeatedly applying $\alpha_i g(\rho_i) = g(\rho_{i+k}) \alpha_i$, we have

$$\begin{aligned}\beta_i \beta_{i-k} \beta_{i-2k} \cdots \beta_{i+2k} \beta_{i+k} &= (\alpha_i g(\rho_i)) (\alpha_{i-k} g(\rho_{i-k})) (\alpha_{i-2k} g(\rho_{i-2k})) \cdots (\alpha_{i+k} g(\rho_{i+k})) \\ &= \alpha_i \alpha_{i-k} \cdots \alpha_{i+2k} \alpha_{i+k} g(\rho_{i+k})^r \\ &= (1_{F_{i+k}} + \rho_{i+k}) (1_{F_{i+k}} + \rho_{i+k})^{-1} \\ &= 1_{F_{i+k}}.\end{aligned}$$

Hence, β_i is an isomorphism for each $i \in \mathbb{Z}_d$ and therefore the morphism β is an isomorphism of matrix factorizations.

We claim $X \cong (\beta_1 \varphi_1, \dots, \varphi_k, \beta_1 \varphi_1, \dots, \varphi_k, \dots, \beta_1 \varphi_1, \dots, \varphi_k)$. For $0 \leq j \leq r-1$ and $2 \leq t \leq k+1$, define $\gamma_{j,t}$ to be the composition of the homomorphisms β_i beginning at F_{t+jk} of length $r-j$. In other words,

$$\gamma_{j,k} = \beta_{t+(r-1)k} \beta_{t+(r-2)k} \cdots \beta_{t+(j+1)k} \beta_{t+jk} : F_{t+jk} \rightarrow F_t.$$

Note that each $\gamma_{j,k}$ is an isomorphism. For $j=0$, the index -1 is interpreted as $r-1$ so that $\gamma_{-1,k+1} = \beta_1$.

Let $0 \leq j \leq r-1$ and $2 \leq t \leq k+1$. To finish the proof, it suffices to show that the following diagram commutes:

$$\begin{array}{ccccccc} F_{k+1+jk} & \xrightarrow{\varphi_{k+jk}} & F_{k+jk} & \xrightarrow{\varphi_{k-1+jk}} & \cdots & \xrightarrow{\varphi_{2+jk}} & F_{2+jk} & \xrightarrow{\varphi_{1+jk}} & F_{1+jk} \\ \downarrow \gamma_{j,k+1} & & \downarrow \gamma_{j,k} & & & & \downarrow \gamma_{j,2} & & \downarrow \gamma_{j-1,k+1} \\ F_{k+1} & \xrightarrow{\varphi_k} & F_k & \xrightarrow{\varphi_{k-1}} & \cdots & \xrightarrow{\varphi_2} & F_2 & \xrightarrow{\beta_1 \varphi_1} & F_{k+1}. \end{array}$$

The commutativity can be broken into three steps. First, we show $\gamma_{j-1,k+1} \varphi_{1+jk} = \beta_1 \varphi_1 \gamma_{j,2}$. Repeatedly applying $\beta_i \varphi_i = \varphi_{i+k} \beta_{i+1}$, $i \in \mathbb{Z}_d$, we have

$$\begin{aligned}
\gamma_{j-1,k+1}\varphi_{1+jk} &= \beta_1\beta_{1-k}\beta_{1-2k} \cdots \beta_{1+(j+1)k}\beta_{1+jk}\varphi_{1+jk} \\
&= \beta_1\varphi_1\beta_{2-k}\beta_{2-2k} \cdots \beta_{2+(j+1)k}\beta_{2+jk} \\
&= \beta_1\varphi_1\gamma_{j,2}.
\end{aligned}$$

Similarly, for $2 \leq t \leq k$, we have

$$\begin{aligned}
\gamma_{j,t}\varphi_{t+jk} &= \beta_{t-k}\beta_{t-2k} \cdots \beta_{t+(j+1)k}\beta_{t+jk}\varphi_{t+jk} \\
&= \varphi_t\beta_{t+1-k}\beta_{t+1-2k} \cdots \beta_{t+1+(j+1)k}\beta_{t+1+jk} \\
&= \varphi_t\gamma_{j,t+1}
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{j,k}\varphi_{k+jk} &= \beta_d\beta_{-k}\beta_{-2k} \cdots \beta_{(j+1)k}\varphi_{(j+1)k} \\
&= \varphi_k\beta_1\beta_{1-k} \cdots \beta_{1+(j+1)k} \\
&= \varphi_k\gamma_{j,k+1}.
\end{aligned}$$

Thus, the d -tuple $\gamma = (\gamma_{-1,k+1}, \gamma_{0,2}, \gamma_{0,3}, \dots, \gamma_{0,k+1}, \gamma_{1,2}, \dots, \gamma_{r-1,k})$ forms an isomorphism from X to $(\beta_1\varphi_1, \dots, \varphi_k, \beta_1\varphi_1, \dots, \varphi_k, \dots, \beta_1\varphi_1, \dots, \varphi_k)$. $\square$

The special case of order 1 will be important going forward.

Corollary 4.3. *Let $X \in \text{MF}_S^d(f)$ be indecomposable of size n and assume $X \cong TX$. Then, there exists a homomorphism $\varphi : S^n \rightarrow S^n$ such that $\varphi^d = f \cdot I_n$ and $X \cong (\varphi, \varphi, \dots, \varphi)$.*

Proposition 4.4. *Let $X \in \text{MF}_S^d(f)$, $N \in \text{MCM}(R^\sharp)$, and assume both X and N are indecomposable objects.*

- (i) *If $X \cong TX$, then $X \cong M^b$ for some $M \in \text{MCM}(R^\sharp)$.*
- (ii) *If $N \cong \sigma^*N$, then $N \cong Y^\sharp$ for some $Y \in \text{MF}_S^d(f)$.*

Proof. If $X \cong TX$, then Corollary 4.3 implies that there exists a free S -module F and an endomorphism $\varphi : F \rightarrow F$ such that $\varphi^d = f \cdot 1_F$ and $X \cong (\varphi, \varphi, \dots, \varphi) \in \text{MF}_S^d(f)$. The pair $(F, \mu^{-1}\varphi)$ defines an $\text{MCM}(R^\sharp)$ module M as follows: As an S -module, $M = F$, and the z -action on M is given by $z \cdot m = \mu^{-1}\varphi(m)$ for all $m \in M$, where $\mu \in S$ satisfies $\mu^d = -1$. Since $(\mu^{-1}\varphi)^d = -f \cdot 1_M$, M is naturally an $R^\sharp$ -module. Since $M = F$ is free over S , it is MCM over $R^\sharp$. By applying $(-)^b$, we have $M^b = (\varphi, \varphi, \dots, \varphi) \cong X$.

Assume $N \cong \sigma^*N$. Using a similar technique to the proof of Proposition 4.2, we obtain an isomorphism of $R^\sharp$ -modules $\theta : N \rightarrow \sigma^*N$ such that

$$(\sigma^{d-1})^*\theta \circ (\sigma^{d-2})^*\theta \circ \dots \circ \sigma^*\theta \circ \theta = 1_N.$$

Such an isomorphism defines the structure of an $R^\sharp[\sigma]$ -module on N . Thus, by [13, Theorem 4.4], there exists $Y \in \text{MF}_S^d(f)$ such that $B(Y) \cong N$ as $R^\sharp[\sigma]$ -modules and therefore $Y^\sharp \cong N$ as $R^\sharp$ -modules by 2.6(i). $\square$

Proposition 4.5. *Let X be indecomposable in $\mathrm{MF}_S^d(f)$ and N be indecomposable in $\mathrm{MCM}(R^\sharp)$.*

- (i) *Assume $X \cong TX$. Then, $X^\sharp \cong \bigoplus_{k \in \mathbb{Z}_d} (\sigma^k)^* M$ for some indecomposable $M \in \mathrm{MCM}(R^\sharp)$ such that $M \not\cong \sigma^* M$.*
- (ii) *The number of indecomposable summands of $X^\sharp$ is at most d . Furthermore, if $X^\sharp$ has exactly d indecomposable summands, then $X \cong TX$.*
- (iii) *Assume $N \cong \sigma^* N$. Then, $N^\flat \cong \bigoplus_{k \in \mathbb{Z}_d} T^k Y$ for some indecomposable $Y \in \mathrm{MF}_S^d(f)$ such that $Y \not\cong TY$.*
- (iv) *The number of indecomposable summands of $N^\flat$ is at most d . Furthermore, if $N^\flat$ has exactly d indecomposable summands, then $N \cong \sigma^* N$.*

Proof. If $X \cong TX$, then 4.4(i) implies $X \cong M^\flat$ for some $M \in \mathrm{MCM}(R^\sharp)$. By Proposition 2.4, we have $X^\sharp \cong M^{\sharp\flat} \cong \bigoplus_{k \in \mathbb{Z}_d} (\sigma^k)^* M$. Similarly, if $N \cong \sigma^* N$, then 4.4(ii) and Proposition 2.4 imply $N^\flat \cong \bigoplus_{k \in \mathbb{Z}_d} T^k Y$ for some $Y \in \mathrm{MF}_S^d(f)$.

Next, in the case that $TX \cong X$, we show that M above is indecomposable and satisfies $M \not\cong \sigma^* M$.

- Suppose $M \cong M_1 \oplus M_2$ for non-zero $M_1, M_2 \in \mathrm{MCM}(R^\sharp)$. Then, $(\sigma^k)^* M \cong (\sigma^k)^* M_1 \oplus (\sigma^k)^* M_2$ for each $k \in \mathbb{Z}_d$. Therefore,

$$X^d \cong X^{\sharp\flat} \cong \bigoplus_{k \in \mathbb{Z}_d} ((\sigma^k)^* M_1)^\flat \oplus ((\sigma^k)^* M_2)^\flat.$$

This contradicts KRS since the left-hand side has precisely d indecomposable summands while the right-hand side has at least $2d$ indecomposable summands. Hence, M is indecomposable.

- Suppose $\sigma^* M \cong M$. Then, since M is indecomposable, the arguments above imply that $M^\flat$ decomposes into a sum of at least d indecomposable summands. Since $T(M^\flat) \cong M^\flat$, we have

$$X^d \cong X^{\sharp\flat} \cong (M^\flat)^{\sharp\flat} \cong (M^\flat)^d.$$

Since X is indecomposable, the left-hand side has precisely d indecomposable summands while the right-hand side has at least d^2 indecomposable summands. Once again, we have a contradiction and so $M \not\cong \sigma^* M$.

This completes the proof of (i). We omit the remaining assertions from (iii) as they follow similarly.

In order to prove (ii), suppose $X^\sharp = M_1 \oplus M_2 \oplus \cdots \oplus M_t$ for non-zero $M_i \in \mathrm{MCM}(R^\sharp)$. Then,

$$X \oplus TX \oplus \cdots \oplus T^{d-1} X \cong X^{\sharp\flat} \cong M_1^\flat \oplus M_2^\flat \oplus \cdots \oplus M_t^\flat. \quad (4.1)$$

The left-hand side has precisely d indecomposable summands and therefore $t \leq d$.

If $X^\sharp$ decomposes into exactly d indecomposables, that is, if $t = d$, then (4.1) implies that $M_i^\flat$ is indecomposable for each i and $X \cong M_j^\flat$ for some $1 \leq j \leq d$. Then, $TX \cong T(M_j^\flat) = M_j^\flat \cong X$.

The proof of (iv) is similar, observing that $\sigma^*(X^\sharp) \cong X^\sharp$ for any $X \in \mathrm{MF}_S^d(f)$. □

5 | REDUCED MATRIX FACTORIZATIONS AND ULRICH MODULES

Let $(S, \mathfrak{n}, \mathbf{k})$ be a complete regular local ring, $0 \neq f \in \mathfrak{n}^2$, and let $d \geq 2$ be an integer. Assume $\mathbf{k}$ is algebraically closed of characteristic not dividing d . In this section, we will consider the following special class of matrix factorizations in $\mathrm{MF}_S^d(f)$.

Definition 5.1. A matrix factorization $X = (\varphi_1, \varphi_2, \dots, \varphi_d) \in \mathrm{MF}_S^d(f)$ is called *reduced* if $\varphi_k : F_{k+1} \rightarrow F_k$ is minimal for each $k \in \mathbb{Z}_d$, that is, if $\mathrm{Im} \varphi_k \subseteq \mathfrak{n}F_k$. Equivalently, after choosing bases, X is reduced if and only if the entries of φ_k lie in $\mathfrak{n}$ for all $k \in \mathbb{Z}_d$. We say that f has *finite reduced d -MF type* if there are, up to isomorphism, only finitely many indecomposable reduced matrix factorizations $X \in \mathrm{MF}_S^d(f)$.

In the case $d = 2$, any indecomposable non-reduced matrix factorization is isomorphic to either $(1, f)$ or $(f, 1)$ in $\mathrm{MF}_S^2(f)$ [14, Remark 7.5]. In particular, this implies that finite 2-MF type is equivalent to finite reduced 2-MF type.

For $d > 2$, the situation is quite different. There are at least as many non-reduced indecomposable d -fold factorizations of f as there are reduced ones [13, Corollary 5.15]. Moreover, finite d -MF type clearly implies finite reduced d -MF type but the converse does not hold for $d > 2$ as we will show in Example 5.11.

Definition 5.2.

- (i) Let $X = (\varphi_1, \dots, \varphi_d) \in \mathrm{MF}_S^d(f)$ and pick bases to consider φ_k , $k \in \mathbb{Z}_d$, as a square matrix with entries in S . Following [4], we define $I(\varphi_k)$ to be the ideal generated by the entries of φ_k and set $I(X) = \sum_{k \in \mathbb{Z}_d} I(\varphi_k)$. Note that the ideal $I(X)$ does not depend on the choice of bases.
- (ii) Let $c_d(f)$ denote the collection of proper ideals I of S such that $f \in I^d$.

In the case $d = 2$, Theorem 3.1 implies that reduced 2-MF type is determined by the cardinality of the set $c_2(f)$. One implication of Theorem 3.1 is proven explicitly in [4]. The authors show that the association $X \mapsto I(X)$ forms a surjection from the set of isomorphism classes of reduced two-fold matrix factorizations of f onto the set $c_2(f)$. Hence, if there are only finitely many indecomposable reduced two-fold matrix factorizations of f up to isomorphism, then the set $c_2(f)$ is finite.

The following result of Herzog, Ulrich, and Backelin shows that the association $X \mapsto I(X)$ remains surjective in the case $d > 2$.

Theorem 5.3 ([9], Theorem 1.2). *Let I be a proper ideal of S and $d \geq 2$. If $f \in I^d$, then there exists a reduced matrix factorization $X \in \mathrm{MF}_S^d(f)$ such that $I(X) = I$.*

Corollary 5.4. *Suppose f has finite reduced d -MF type. Then, $c_d(f)$ is a finite collection of ideals of S .*

Corollary 5.4 extends one direction of Theorem 3.1; however, the converse does not hold for $d > 2$ as shown by the next example.

Example 5.5. Let $S = \mathbf{k}[[x, y]]$ with $\mathrm{char} \mathbf{k} \neq 2$ and $f = x^2y \in S$. Then, the one-dimensional D_∞ singularity $R = S/(f)$ has countably infinite CM type by [4, Proposition 4.2]. For each $k \geq 1$, we

have a reduced matrix factorization of x^2y with three factors:

$$X_k = \left(\begin{pmatrix} x & y^k \\ 0 & -x \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} x & y^k \\ 0 & -x \end{pmatrix} \right) \in \text{MF}_S^3(x^2y).$$

Any isomorphism $X_k \rightarrow X_j$ for $k, j \geq 1$ induces an isomorphism of R -modules $\text{cok} \begin{pmatrix} x & y^k \\ 0 & -x \end{pmatrix} \rightarrow \text{cok} \begin{pmatrix} x & y^j \\ 0 & -x \end{pmatrix}$. Such an isomorphism is only possible if $k = j$, that is, $X_k \cong X_j$ if and only if $k = j$. Since X_k is reduced and the MCM R -module $\text{cok} \begin{pmatrix} x & y^k \\ 0 & -x \end{pmatrix}$ is indecomposable, [13, Lemma 6.6] implies that X_k is indecomposable. Thus, x^2y has infinite reduced 3-MF type.

On the other hand, it is not hard to see that $c_3(x^2y)$ contains only the maximal ideal. So $c_3(x^2y)$ is a finite set but x^2y has infinite reduced 3-MF type.

Let N be an MCM $R^\sharp$ -module and let $\mu_{R^\sharp}(N)$ denote the minimal number of generators of N . Recall that N is finitely generated and free over S . We will see below that there is an inequality

$$\mu_{R^\sharp}(N) \leq \text{rank}_S(N). \quad (5.1)$$

In the following, we consider MCM $R^\sharp$ -modules N where the equality $\mu_{R^\sharp}(N) = \text{rank}_S(N)$ is attained.

As we saw in Example 3.4, a matrix factorization of the form N^b , obtained by computing multiplication by z on an MCM $R^\sharp$ -module N , can be non-reduced. We will show below that the matrix representing multiplication by z on N contains unit entries precisely when $\mu_{R^\sharp}(N) < \text{rank}_S(N)$. In other words, the restriction of the functor $(-)^b : \text{MCM}(R^\sharp) \rightarrow \text{MF}_S^d(f)$ to the subcategory of MCM $R^\sharp$ -modules satisfying $\mu_{R^\sharp}(N) = \text{rank}_S(N)$ produces only reduced matrix factorizations of f with d factors. Conversely, the image of the functor $(-)^{\sharp} : \text{MF}_S^d(f) \rightarrow \text{MCM}(R^\sharp)$, restricted to the subcategory of reduced matrix factorizations of f , consists exactly of the MCM $R^\sharp$ -modules N satisfying $\mu_{R^\sharp}(N) = \text{rank}_S(N)$.

Lemma 5.6. *Let N be an MCM $R^\sharp$ -module and assume $f + z^d$ is irreducible. Then, N is a finitely generated free S -module satisfying*

$$\mu_{R^\sharp}(N) \leq \text{rank}_S(N) = d \cdot \text{rank}_{R^\sharp}(N) = \text{rank}_S(R^\sharp) \cdot \text{rank}_{R^\sharp}(N).$$

Proof. Let $(\Phi : S[[z]]^n \rightarrow S[[z]]^n, \Psi : S[[z]]^n \rightarrow S[[z]]^n) \in \text{MF}_{S[[z]]}^2(f + z^d)$ be a matrix factorization of $f + z^d$ such that Φ is minimal and $\text{cok } \Phi = N$. Since Φ is minimal, $n = \mu_{R^\sharp}(N)$. Then, $\det \Phi = u(f + z^d)^k$ for some $1 \leq k \leq n$ and some unit $u \in S[[z]]$. Recall that $k = \text{rank}_{R^\sharp}(N)$ by [6, Proposition 5.6]. By tensoring with $S = S[[z]]/(z)$, we find $\det \bar{\Phi} = v \cdot f^k$, where $\bar{\Phi} = \Phi \otimes_{S[[z]]} 1_S$ and $v \in S$ is a unit. Moreover, $\bar{\Phi}$ is injective since $\bar{\Phi}\bar{\Psi} = f \cdot 1_{S^n} = \bar{\Psi}\bar{\Phi}$, and we have a minimal presentation of N/zN over S :

$$0 \longrightarrow S^n \xrightarrow{\Phi} S^n \longrightarrow N/zN \longrightarrow 0.$$

On the other hand, since N is MCM over $R^\sharp$, it is finitely generated and free as an S -module. Let $r = \text{rank}_S(N)$ and consider the map $\varphi : S^r \rightarrow S^r$ representing multiplication by z on N . This

map also gives a presentation of N/zN over S , though the presentation may not be minimal (see Example 3.4). Thus, there exists a commutative diagram with vertical isomorphisms

$$\begin{array}{ccccccc}
 0 & \longrightarrow & S^r & \xrightarrow{\begin{pmatrix} \Phi & 0 \\ 0 & I_{r-n} \end{pmatrix}} & S^r & \longrightarrow & N/zN \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & S^r & \xrightarrow{\varphi} & S^r & \longrightarrow & N/zN \longrightarrow 0.
 \end{array} \quad (5.2)$$

This implies $\mu_R(N/zN) \leq r$, where $R = S/(f)$ as usual. The desired inequality now follows from the fact that $\mu_R(N/zN) = \mu_{R^\#}(N)$. Furthermore, the diagram implies $\det \varphi = v' \cdot f^k$ for some unit v' . However, since $\varphi^d = -f \cdot I_r$, we have that, up to units, $f^r = (\det \varphi)^d = f^{kd}$. Thus, $\text{rank}_S(N) = r = dk = d \cdot \text{rank}_{R^\#}(N)$. $\square$

Lemma 5.7. *Assume $f + z^d$ is irreducible. Let N be an MCM $R^\#$ -module and let $X \in \text{MF}_S^d(f)$. Then, $\mu_{R^\#}(N) = \text{rank}_S(N)$ if and only if $N^b \in \text{MF}_S^d(f)$ is reduced, and $X^\#$ satisfies $\mu_{R^\#}(X^\#) = \text{rank}_S(X^\#)$ if and only if X is reduced.*

Proof. Let $N \in \text{MCM}(R^\#)$ and set $r = \text{rank}_S(N)$. Let $\varphi : S^r \rightarrow S^r$ be the S -linear map representing multiplication by z on N . Then, the presentation of N/zN given by φ in (5.2) is minimal if and only if $r = \text{rank}_S(N) = \mu_R(N/zN)$, where $R = S/(f)$. Since $\mu_R(N/zN) = \mu_{R^\#}(N)$, we have that φ is minimal if and only if $\text{rank}_S(N) = \mu_{R^\#}(N)$. This proves the first statement since $N^b = (\mu\varphi, \mu\varphi, \dots, \mu\varphi) \in \text{MF}_S^d(f)$.

By Proposition 2.4, $X^\# \cong \bigoplus_{k \in \mathbb{Z}_d} T^k X$ which is reduced if and only if X is reduced. The second statement now follows from the first by taking $N = X^\# \in \text{MCM}(R^\#)$. $\square$

Lemma 5.7 gives us a specialization of Corollary 2.5 and Theorem A.

Proposition 5.8. *Assume $f + z^d$ is irreducible.*

- (i) *For any reduced $X \in \text{MF}_S^d(f)$, there exists $N \in \text{MCM}(R^\#)$ satisfying $\text{rank}_S(N) = \mu_{R^\#}(N)$ such that X is isomorphic to a direct summand of N^b .*
- (ii) *For any $N \in \text{MCM}(R^\#)$ satisfying $\text{rank}_S(N) = \mu_{R^\#}(N)$, there exists reduced $X \in \text{MF}_S^d(f)$ such that N is isomorphic to a direct summand of $X^\#$.*

In particular, f has finite reduced d -MF type if and only if there are, up to isomorphism, only finitely many indecomposable MCM $R^\#$ -modules N satisfying $\text{rank}_S(N) = \mu_{R^\#}(N)$.

Proof. Both (i) and (ii) follow from Lemma 5.7 and Proposition 2.4. The final statement follows as in the proof of Theorem A by noticing that a matrix factorization $Y \in \text{MF}_S^d(f)$ is reduced if and only if every summand of Y is reduced and that an MCM $R^\#$ -module N satisfies $\mu_{R^\#}(N) = \text{rank}_S(N)$ if and only if every summand of N satisfies the same equality. $\square$

For a module M over a local ring A , we let $e(M)$ denote the multiplicity of M . If M is an MCM A -module, there is a well-known inequality $\mu_A(M) \leq e(M)$. The class of MCM modules satisfying $\mu_A(M) = e(M)$ are called *Ulrich modules*. For background on Ulrich modules, we refer the reader

to [2, 3, 8] and [9]. If A is a domain, then we may compute the multiplicity of M as $e(M) = e(A) \cdot \text{rank}_A(M)$ (see [14, Proposition 1.6]).

In the case of the d -fold branched cover of R , we have the following connection between reduced d -fold matrix factorizations of f and Ulrich modules over $R^\sharp$.

Corollary 5.9. *Assume $d \leq \text{ord}(f)$ and $f + z^d$ is irreducible. Let $N \in \text{MCM}(R^\sharp)$. Then, N is an Ulrich $R^\sharp$ -module if and only if $N^\flat \in \text{MF}_S^d(f)$ is a reduced matrix factorization.*

In particular, f has finite reduced d -MF type if and only if there are, up to isomorphism, only finitely many indecomposable Ulrich $R^\sharp$ -modules.

Proof. Since $d \leq \text{ord}(f)$, the multiplicity of $R^\sharp = S[[z]]/(f + z^d)$ is d . Hence, an MCM $R^\sharp$ -module N is Ulrich if and only if $\mu_{R^\sharp}(N) = d \cdot \text{rank}_{R^\sharp}(N)$. By Lemma 5.6, the quantity $d \cdot \text{rank}_{R^\sharp}(N)$ is equal to the rank of N as a free S -module. Thus, N is Ulrich if and only if $\mu_{R^\sharp}(N) = \text{rank}_S(N)$. Both statements now follow from Proposition 5.8. $\square$

Remark 5.10. Let $d = 2$ so that $R^\sharp = S[[z]]/(f + z^2)$ is the double branched cover. The condition $\text{rank}_S(N) = \mu_{R^\sharp}(N)$ is redundant in this case. An MCM $R^\sharp$ -module N satisfies $\text{rank}_S(N) = \mu_{R^\sharp}(N)$ if and only if N has no summands isomorphic to $R^\sharp$ (this follows from the proof of [11, Lemma 8.17(iii)]). In other words, the conclusion of Proposition 5.8 is simply a restatement of Knörrer's Theorem (Theorem 1.3 above) when $d = 2$. Furthermore, Corollary 5.9 implies that any MCM $R^\sharp$ -module with no free summands is an Ulrich module. This is a known result of Herzog–Kühl [8, Corollary 1.4] since the multiplicity of $R^\sharp$ is two.

Example 5.11. Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and consider the one-dimensional hypersurface ring

$$R_{a,i} = \mathbf{k}[[x, y]]/(x^a + y^{a+i}), \quad a \geq 2, i \geq 0.$$

If $i = 1$ or $i = 2$, then, by [9, Theorem A.3], $R_{a,i}$ has only finitely many isomorphism classes of indecomposable Ulrich modules. Set $S = \mathbf{k}[[y]]$ and consider $R_{a,i}$ as the a -fold branched cover of $R = \mathbf{k}[[y]]/(y^{a+i})$. Since $e(R_{a,i}) = a$, Corollary 5.9 implies that y^{a+i} , for $i \in \{1, 2\}$, has only finitely many isomorphism classes of reduced indecomposable a -fold matrix factorizations. In other words, y^{a+i} has finite reduced a -MF type for $i = 1, 2$ and any $a \geq 2$.

The methods in [9, Theorem A.3] can be used to compute the isomorphism classes of indecomposable reduced matrix factorizations of y^{a+i} . For instance, let $a \geq 2$ and $i = 1$. Then, $R_{a,1} \cong \mathbf{k}[[t^a, t^{a+1}]]$ and t^a is a minimal reduction of the maximal ideal $\mathfrak{m}$ of $R_{a,1}$. Hence, $R'_{a,1} = R_{a,1}[\frac{r}{t^a} : r \in \mathfrak{m}] = \mathbf{k}[[t]]$ is the first quadratic transform of $R_{a,1}$. By [9, Corollary A.1], an $R_{a,1}$ -module M is Ulrich over $R_{a,1}$ if and only if it is MCM over $R'_{a,1}$. Since $R'_{a,1} = \mathbf{k}[[t]]$ is a regular local ring, the only indecomposable MCM $R'_{a,1}$ -module is $R'_{a,1}$ itself.

As an $S \cong \mathbf{k}[[t^a]]$ -module, $R'_{a,1} = \mathbf{k}[[t]]$ is free with basis given by $\{1, t, t^2, \dots, t^{a-1}\}$. Thus, multiplication by $x = t^{a+1}$ on the basis $\{1, t, \dots, t^{a-1}\}$ is given by the mapping

$$t^k \mapsto t^{a+1+k} = t^a t^{k+1}$$

for $0 \leq k \leq a-1$. Since $y = t^a$, it follows that multiplication by x on the MCM $R_{a,1}$ -module $R'_{a,1}$ is given by the $a \times a$ matrix with entries in $\mathbf{k}[[y]]$

$$\varphi = \begin{pmatrix} 0 & 0 & \cdots & 0 & y^2 \\ y & 0 & \cdots & 0 & 0 \\ 0 & y & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & y & 0 \end{pmatrix}.$$

It follows that $(R'_{a,1})^b \cong \bigoplus_{i \in \mathbb{Z}_a} T^i((y^2, y, y, \dots, y)) \in \text{MF}_{\mathbf{k}[[y]]}^a(y^{a+1})$. By Proposition 5.8 and Corollary 5.9, the matrix factorization $(y^2, y, y, \dots, y) \in \text{MF}_{\mathbf{k}[[y]]}^a(y^{a+1})$, and its corresponding shifts, are the only indecomposable reduced matrix factorizations of y^{a+1} with a factors.

Notice that for $a \geq 4$, the polynomial y^{a+1} does not appear on the list given in Theorem B for any $d > 2$. Thus, the conclusions of this example imply that, for $a \geq 4$, the polynomial y^{a+1} has infinitely many isomorphism classes of indecomposable matrix factorizations with a factors but only finitely many which are reduced.

The last example shows the necessity of the assumption $d \leq \text{ord}(f)$ in Corollary 5.9.

Example 5.12. Let $\mathbf{k}$ be algebraically closed of characteristic zero. Set $S = \mathbf{k}[[x]]$, $f = x^3$, and $R = S/(f)$. The hypersurface ring $R^\sharp = \mathbf{k}[[x, y]]/(x^3 + y^4)$ is the same ring given in Example 3.4, however, here we are viewing $R^\sharp$ as the four-fold branched cover of $R = \mathbf{k}[[x]]/(x^3)$. Again using the notation of [14, 9.13], we take $B = \text{cok } \beta$ where

$$\beta = \begin{pmatrix} y & 0 & x \\ x & -y^2 & 0 \\ 0 & x & -y \end{pmatrix}.$$

The MCM $R^\sharp$ -module B is, in this case, free of rank 4 over $S = \mathbf{k}[[x]]$. In particular, if $e_1, e_2, e_3 \in B$ are the images of the standard basis on $S[[y]]^3$, then an S -basis for B is $\{e_1, e_2, e_3, ye_2\}$. Multiplication by y on B is therefore given by the S -matrix

$$\varphi = \begin{pmatrix} 0 & 0 & x & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Notice that B is an Ulrich $R^\sharp$ -module but multiplication by y on B is given by a non-reduced matrix. In other words, the condition $d \leq \text{ord}(f)$ in Proposition 5.8 is needed to produce reduced matrix factorizations of f .

ACKNOWLEDGMENTS

The first author was supported by a CUSE grant from Syracuse University.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and

mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Graham J. Leuschke  <https://orcid.org/0000-0002-9408-0253>

Tim Tribone  <https://orcid.org/0000-0002-9981-6995>

REFERENCES

1. V. I. Arnol'd, *Normal forms for functions near degenerate critical points, the Weyl groups of A_k , D_k , E_k and Lagrangian singularities*, Funct. Anal. Appl. **6** (1973), no. 4, 254–272 (en).
2. A. Beauville, *An introduction to Ulrich bundles*, Eur. J. Math. **4** (2018), no. 1, 26–36 (en).
3. J. P. Brennan, J. Herzog, and B. Ulrich, *Maximally generated Cohen-Macaulay modules*, Math. Scand. **61** (1987), no. 2, 181–203. MR 947472.
4. R.-O. Buchweitz, G.-M. Greuel, and F.-O. Schreyer, *Cohen-Macaulay modules on hypersurface singularities. II*, Invent. Math. **88** (1987), no. 1, 165–182. MR 877011.
5. R.-O. Buchweitz and A. Pavlov, *Moore matrices and Ulrich bundles on an elliptic curve*, 2015, <https://arxiv.org/abs/1511.05502>.
6. D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 35–64. MR 570778.
7. P. Gabriel, *The universal cover of a representation-finite algebra*, Representations of Algebras (Maurice Auslander and Emilo Lluís, eds.), Springer, Berlin, Heidelberg, 1981, pp. 68–105.
8. J. Herzog and M. Kühl, *Maximal Cohen-Macaulay modules over Gorenstein rings and Bourbaki-sequences*, Commutative Algebra and Combinatorics (Kyoto, 1985), Adv. Stud. Pure Math., vol. 11, North-Holland, Amsterdam, 1987, pp. 65–92. MR 951197.
9. J. Herzog, B. Ulrich, and J. Backelin, *Linear maximal Cohen-Macaulay modules over strict complete intersections*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 187–202.
10. H. Knörrer, *Cohen-Macaulay modules on hypersurface singularities. I*, Invent. Math. **88** (1987), no. 1, 153–164. MR 877010.
11. G. J. Leuschke and R. Wiegand, *Cohen-Macaulay representations*, Mathematical Surveys and Monographs, vol. 181, American Mathematical Society, Providence, RI, 2012. MR 2919145.
12. I. Reiten and C. Riedtmann, *Skew group algebras in the representation theory of Artin algebras*, J. Algebra **92** (1985), no. 1, 224–282 (en).
13. T. Tribone, *Matrix factorizations with more than two factors*, 2021, <https://arxiv.org/abs/2102.06819>.
14. Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, vol. 146, Cambridge University Press, Cambridge, 1990. MR 1079937.